Calculus MA1001-B Quiz 9

National Central University, Nov. 26 2019

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Problem 1. (3pts) Use the definition of ln to show that $\ln(ab) = \ln a + \ln b$ for all $a, b > 0$.

Proof. Let $a, b > 0$ be given. By definition,

$$\ln(ab) = \int_1^{ab} \frac{dx}{x} = \int_1^a \frac{dx}{x} + \int_a^{ab} \frac{dx}{x} = \ln a + \int_a^{ab} \frac{dx}{x}\,.$$

Using the substitution of variable $ax = u$ $\big($which implies that $adx = du$ or $dx = \frac{1}{a}du\big)$, we find that

$$\int_a^{ab} \frac{dx}{x} = \int_1^b \frac{1}{a}\frac{du}{u/a} = \int_1^b \frac{du}{u} = \ln b\,;$$

thus $\ln(ab) = \ln a + \ln b$. □

Problem 2. (4pts) Find the derivative of the function $y = \dfrac{\cos^2(1+\ln x)}{x^2}$ by first taking the logarithm (base e) and then differentiating.

Solution. Let $f(x) = \dfrac{\cos^2(1+\ln x)}{x^2}$. Taking the logarithm,

$$\ln f(x) = \ln\cos^2(1+\ln x) - \ln(x^2) = 2\ln\big|\cos(1+\ln x)\big| - 2\ln|x|\,;$$

thus if $\cos\ln(1+x) \neq 0$ and $x \neq 0$,

$$\frac{d}{dx}\ln f(x) = \frac{-2\sin(1+\ln x)}{\cos(1+\ln x)}\cdot\frac{1}{x} - \frac{2}{x} = -\frac{2\sin(1+\ln x)}{x\cos(1+\ln x)} - \frac{2}{x}\,.$$

Since $\dfrac{d}{dx}\ln f(x) = \dfrac{f'(x)}{f(x)}$, we find that

$$\begin{aligned} f'(x) = f(x)\frac{d}{dx}\ln f(x) &= \frac{\cos^2(1+\ln x)}{x^2}\Big[-\frac{2\sin(1+\ln x)}{x\cos(1+\ln x)} - \frac{2}{x}\Big] \\ &= -\frac{2\sin(1+\ln x)\cos(1+\ln x) + 2\cos^2(1+\ln x)}{x^3}\,. \end{aligned}$$

□

Problem 3. (3pts) Find the indefinite integral $\displaystyle\int \frac{\sin^3(1+\ln x)}{x}\,dx$.

Solution. Let $u = 1 + \ln x$. Then $du = \dfrac{dx}{x}$; thus

$$\begin{aligned} \int \frac{\sin^3(1+\ln x)}{x}\,dx = \int \sin^3 u\,du = \int \frac{3\sin u - \sin(3u)}{4}\,du &= \frac{1}{2}\cos(3u) - \frac{3}{4}\cos u + C \\ &= \frac{1}{2}\cos\big[3(1+\ln x)\big] - \frac{3}{4}\cos(1+\ln x) + C\,. \end{aligned}$$

□