# Calculus MA1001-B Quiz 8

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**Problem 1.** (2pts) State the Fundamental Theorem of Calculus.

*Solution. Theorem*: Let $f : [a,b] \to \mathbb{R}$ be a continuous functions, and $F$ be an anti-derivative of $f$ on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Moreover, if $G(x) = \displaystyle\int_a^x f(t)\,dt$, then $G$ is an anti-derivative of $f$ on $[a,b]$. □

**Problem 2.** (3pts) Find the definite integral $\displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$.

*Solution.* Let $u = 1+\sqrt{x}$. Then $x = (u-1)^2$; thus $dx = 2(u-1)du$. By the substitution of variable,

$$\begin{aligned}\int_0^1 \frac{dx}{(1+\sqrt{x})^4} &= \int_1^2 \frac{2(u-1)}{u^4}\,du = 2\int_1^2 \left(u^{-3} - u^{-4}\right)du = 2\Big(\frac{1}{3}u^{-3} - \frac{1}{2}u^{-2}\Big)\Big|_{u=1}^{u=2} \\ &= 2\Big(\frac{1}{3}\cdot\frac{1}{8} - \frac{1}{2}\cdot\frac{1}{4}\Big) - 2\Big(\frac{1}{3} - \frac{1}{2}\Big) = \frac{1}{6}\,.\end{aligned}$$

□

**Problem 3.** (3pts) Let $f : [-1,1] \to \mathbb{R}$ be a continuous function. Show that

$$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx\,.$$

*Proof.* Let $u = \pi - x$. Then $du = -dx$. By the substitution of variable,

$$\int_0^\pi x f(\sin x)\,dx = \int_\pi^0 (\pi - u) f(\sin u)(-du) = \int_0^\pi f(\sin u)\,du - \int_0^\pi u f(\sin u)\,du\,;$$

thus by the fact that $\displaystyle\int_0^\pi u f(\sin u)\,du = \int_0^\pi x f(\sin x)\,dx$, we conclude that

$$\int_0^\pi x f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx\,.$$

□

**Problem 4.** (2pts) Find, explain, and correct the mistake on the following computation of integral.

$$\int_1^{-1} \frac{dx}{1+x^2} \overset{(x=\tan u)}{=} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\sec^2 u du}{1+\tan^2 u} = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} du = \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}\,.$$

*Solution.* In order to apply the substitution of variable formula

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u))g'(u)\,du\,,$$

it is requiremed that the function $g$ is continuously differentiable on the interval $[a,b]$ or $[b,a]$; however, tan is not differentiable at $\frac{\pi}{2} \in [\frac{\pi}{4}, \frac{3\pi}{4}]$, so the application of the substitution of variable is wrong. Instead, with the same substitution of variable,

$$\int_1^{-1} \frac{dx}{1+x^2} \overset{(x=\tan u)}{=} \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{\sec^2 u du}{1+\tan^2 u} = \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} du = \frac{-\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}\,.$$

□