Calculus MA1001-B Quiz 7

National Central University, Nov. 12 2019

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Problem 1. (2pts) State the Mean Value Theorem for integrals.

Solution. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx\,. \qquad \square$$

Problem 2. (6pts) State and prove the Fundamental Theorem of Calculus.

Solution. Theorem: Let $f : [a, b] \to \mathbb{R}$ be a continuous functions, and F be an anti-derivative of f on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Moreover, if $G(x) = \int_a^x f(t)\,dt$, then G is an anti-derivative of f on $[a, b]$.

Proof: Let $G(x) = \int_a^x f(t)\,dt$. Then for $h \neq 0$ such that $x + h \in [a, b]$, we have

$$\frac{G(x+h) - G(x)}{h} = \frac{1}{h}\Big[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\Big] = \frac{1}{h}\int_x^{x+h} f(t)\,dt\,.$$

By the mean value theorem for integrals, there exists c between x and $x + h$ such that

$$\frac{G(x+h) - G(x)}{h} = f(c)\,.$$

Since $c \to x$ as $h \to 0$, by the continuity of f we find that $\lim\limits_{h\to 0} f(c) = \lim\limits_{c\to x} f(c) = f(x)$. Therefore,

$$G'(x) = \lim_{h\to 0} \frac{G(x+h) - G(x)}{h} = f(x)$$

which shows that G is an anti-derivative of f.

Since F is an anti-derivative of G, by the mean value theorem there exists C such that $G(x) = F(x) + C$ for all $x \in [a, b]$. Therefore, by the fact $G(a) = 0$, we find that $C = G(a) - F(a) = -F(a)$; thus $G(x) = F(x) - F(a)$. As a consequence,

$$\int_a^b f(x)\,dx = G(b) = F(b) - F(a)\,. \qquad \square$$

Problem 3. (2pts) Use the Fundamental Theorem of Calculus to find the integral of $\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\,dx$.

Solution. Since $\frac{d}{dx}(-\cos x) = \sin x$, we find that $y = -\cos x$ is an anti-derivative of $y = \sin x$; thus the Fundamental Theorem of Calculus implies that

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x\,dx = \big(-\cos\frac{\pi}{3}\big) - \big(-\cos(-\frac{\pi}{6})\big) = \frac{1}{2}(\sqrt{3} - 1)\,. \qquad \square$$