Calculus MA1001-B Quiz 4

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Problem 1. (2%) Let f be a function defined on an open interval I containing c. Show that if f is differentiable at c, then f is continuous at c.

Proof. Suppose that f is differentiable at c. Then $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c} = f'(c)$ exists. Since $\lim\limits_{x\to c}(x-c)=0$ also exists, we must have

$$\lim_{x\to c}\big[f(x)-f(c)\big] = \lim_{x\to c}\Big(\frac{f(x)-f(c)}{x-c}\cdot(x-c)\Big) = \Big(\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\Big)\Big(\lim_{x\to c}(x-c)\Big) = f'(c)\cdot 0 = 0\,;$$

thus $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c}\big[f(x)-f(c)\big] + \lim\limits_{x\to c} f(c) = f(c)$ which shows that f is continuous at c. □

Problem 2. (4%) Let $f, g : \mathbb{R}\to\mathbb{R}$ be twice differentiable functions. Show that if $y = f(u)$ and $u = g(x)$, then

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\Big(\frac{du}{dx}\Big)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}\,.$$

Proof. By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}\,;$$

thus the product rule further implies that

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\Big(\frac{dy}{du}\frac{du}{dx}\Big) = \Big(\frac{d}{dx}\frac{dy}{du}\Big)\frac{du}{dx} + \frac{dy}{du}\Big(\frac{d}{dx}\frac{du}{dx}\Big) = \Big(\frac{d^2y}{du^2}\frac{du}{dx}\Big)\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2}\\ &= \frac{d^2y}{du^2}\Big(\frac{du}{dx}\Big)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}\,.\end{aligned}$$

□

Problem 3. (4%) The equation $x^2 - xy + y^2 = 3$ represents a "rotated ellipse"; that is, an ellipse whose axes are not parallel to the coordinate axes. Find the points at which this ellipse crosses the x-axis and show that the tangent lines at these points are parallel.

Solution. First we solves for the x-intercepts of the ellipse. Suppose that the ellipse crosses the x-axis at $(x, 0)$. Then

$$x^2 - x\cdot 0 + 0^2 = 3$$

which shows that the ellipse crosses the x-axis at $(\pm\sqrt{3}, 0)$. By implicit differentiation, the slope $\dfrac{dy}{dx}$ of the tangent line of the ellipse at (x, y) satisfies that

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

which implies that $\dfrac{dy}{dx} = \dfrac{2x-y}{x-2y}$ if $x \neq 2y$. In particular,

$$\frac{dy}{dx}\Big|_{(x,y)=(\sqrt{3},0)} = 2 \qquad \text{and} \qquad \frac{dy}{dx}\Big|_{(x,y)=(-\sqrt{3},0)} = 2\,;$$

thus the tangent lines at $(\pm\sqrt{3}, 0)$ are parallel (since they have the same slope). □