Calculus MA1001-B Quiz 1

National Central University, Sept. 24 2019

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Problem 1. (4%) Let f be a function defined on an open interval containing c (except possibly at c). State the definition of $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$. Do **NOT** use logic symbols.

Solution. 1. $\lim_{x\to c} f(x) = L$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\big|f(x) - L\big| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta\,.$$

2. $\lim_{x\to c^-} f(x) = L$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\big|f(x) - L\big| < \varepsilon \quad \text{whenever} \quad c - \delta < x < c\,.$$

□

Problem 2. (3%) Find the limit $\lim_{x\to 0} x\sqrt{1 + \frac{1}{x^2}}$.

Solution. For $x \neq 0$, $x\sqrt{1 + \frac{1}{x^2}} = x\sqrt{\frac{1+x^2}{x^2}} = x\frac{\sqrt{1+x^2}}{\sqrt{x^2}} = \frac{x}{|x|}\sqrt{1+x^2}$. Therefore,

$$\lim_{x\to 0^+} x\sqrt{1 + \frac{1}{x^2}} = \lim_{x\to 0^+} \frac{x}{x}\sqrt{1+x^2} = \lim_{x\to 0^+} \sqrt{1+x^2} = 1$$

and

$$\lim_{x\to 0^-} x\sqrt{1 + \frac{1}{x^2}} = \lim_{x\to 0^-} \frac{x}{-x}\sqrt{1+x^2} = -\lim_{x\to 0^-} \sqrt{1+x^2} = -1\,.$$

Since $\lim_{x\to 0^+} x\sqrt{1 + \frac{1}{x^2}} \neq \lim_{x\to 0^-} x\sqrt{1 + \frac{1}{x^2}}$, we find that $\lim_{x\to 0} x\sqrt{1 + \frac{1}{x^2}}$ D.N.E. □

Problem 3. (3%) Let $-\frac{\pi}{2} < c < \frac{\pi}{2}$. Find the limit $\lim_{x\to c} \frac{\tan x - \tan c}{x - c}$ using the identities

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \qquad \text{and} \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1\,.$$

Solution. Using the identity above,

$$\tan x - \tan c = (1 + \tan x \tan c)\tan(x - c) \qquad \forall -\frac{\pi}{2} < x < \frac{\pi}{2}\,;$$

thus

$$\frac{\tan x - \tan c}{x - c} = \frac{\sin(x-c)}{x - c}\frac{1 + \tan x \tan c}{\cos(x - c)} \qquad \forall\, x \neq c \text{ and } -\frac{\pi}{2} < x < \frac{\pi}{2}\,.$$

Using the identity $\lim_{x\to 0} \frac{\sin x}{x} = 1$, we find that

$$\lim_{x\to c} \frac{\sin(x-c)}{x-c} = 1$$

Therefore, by the fact that $\lim_{x\to 0} \frac{1 + \tan x \tan c}{\cos(x - c)} = 1 + \tan^2 c = \sec^2 c$, we find that

$$\lim_{x\to c} \frac{\tan x - \tan c}{x - c} = \Big(\lim_{x\to c} \frac{\sin(x-c)}{x-c}\Big)\Big(\lim_{x\to c} \frac{1 + \tan x \tan c}{\cos(x-c)}\Big) = \sec^2 c\,.$$

□