Exercise Problem Sets 7

Nov. 1. 2019

Problem 1. Let $f : [a, b] \to \mathbb{R}$ be a function, and f is Riemann integrable on $[a, b]$. Show that f must be bounded on $[a, b]$; that is, there exists a real number $M > 0$ such that $|f(x)| \leqslant M$ for all $a \leqslant x \leqslant b$.

Problem 2. Let $a < b$ be real numbers. Compute $\displaystyle\int_a^b \cos x \, dx$ by the following steps.

(a) Partition $[a, b]$ into n sub-intervals with equal length. Write down the Riemann sum using the right end-point rule.

(b) Prove that

$$\sum_{i=1}^{n} \cos(a + id) = \frac{\sin\left[a + \left(n + \frac{1}{2}\right)d\right] - \sin\left(a + \frac{d}{2}\right)}{2 \sin \frac{d}{2}} . \qquad (\star)$$

Hint: Use the sum and difference formula $\sin(\vartheta + \varphi) - \sin(\vartheta - \varphi) = 2 \sin \vartheta \cos \varphi$.

(c) Use $(\star)$ to simplify the Riemann sum in (a), and find the limit of the Riemann sum as n approaches infinity. Show that

$$\int_a^b \cos x \, dx = \sin b - \sin a \, .$$

Problem 3. Let $a < b$ be real numbers. Compute $\displaystyle\int_a^b x^N \, dx$, where N is a non-negative integer, by the following steps.

(a) Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a regular partition of $[a, b]$. Show that the Riemann sum using the right end-point rule is given by

$$I_n = \sum_{k=0}^{N} \left[C_k^N a^{N-k} (b - a)^{k+1} \left(\frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k \right) \right] ,$$

where $C_k^N = \dfrac{N!}{k!(N-k)!}$.

(b) Show that

$$\sum_{i=1}^{n} i^k = \frac{1}{k+1}(n+1)^{k+1} - \frac{1}{k+1} \left[C_{k-1}^{k+1} \sum_{i=1}^{n} i^{k-1} + \cdots + C_1^{k+1} \sum_{i=1}^{n} i + (n+1) \right] . \qquad (\star\star)$$

Hint: Expand $(j + 1)^k$ for $j = 0, 1, 2, \cdots, n$ by the binomial expansion formula, and sum over j to obtain the equality above.

(c) Use $(\star\star)$ to show that $\displaystyle\lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{i=1}^{n} i^k = \frac{1}{k+1}$ for each $k \in \mathbb{N}$.

(d) Use the limit in (c) to find the limit of the Riemann sum in (a) by passing to the limit as n approaches infinity. Simplify the result to show that

$$\int_a^b x^N\,dx = \frac{b^{N+1} - a^{N+1}}{N+1}\,.$$

Hint: (c) By induction!

Problem 4. In class we have used the limit of Riemann sums to compute the integral $\displaystyle\int_0^\pi x\cos x\,dx$. Find this integral by completing what we did in class.

Problem 5. Determine the following limits by identifying the limits as limits of certain Riemann sums so that the limits are the same as certain integrals.

1. $\displaystyle\lim_{n\to\infty} \frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{n}}{n^{\frac{3}{2}}}$.

2. $\displaystyle\lim_{n\to\infty} \frac{1}{\sqrt{n}}\Big(1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\Big)$.

3. $\displaystyle\lim_{n\to\infty} \Big[\frac{1}{\sqrt{n^2+2n}}+\frac{1}{\sqrt{n^2+4n}}+\frac{1}{\sqrt{n^2+6n}}+\cdots+\frac{1}{\sqrt{n^2+2n^2}}\Big]$.

Problem 6. Let $f:[a,b]\to\mathbb{R}$ be Riemann integrable on $[a,b]$, and $m\leqslant f(x)\leqslant M$ for all $x\in[a,b]$. Show that

$$m(b-a)\leqslant \int_a^b f(x)\,dx \leqslant M(b-a)\,.$$

Problem 7. Let $f:[0,1]\to\mathbb{R}$ be a function satisfying that

$$\big|f(x)-f(y)\big| \leqslant M|x-y| \qquad \forall\, x,y\in[0,1]\,.$$

Under the fact that f is Riemann integrable on $[0,1]$, show that

$$\Big|\int_0^1 f(x)\,dx - \frac{1}{n}\sum_{i=1}^n f\big(\frac{i}{n}\big)\Big| < \frac{M}{2n}\,.$$

Problem 8. Suppose that $f,g:[a,b]\to\mathbb{R}$ are Riemann integrable on $[a,b]$. Under the fact that fg is Riemann integrable on $[a,b]$, show that

$$\int_a^b f(x)g(x)\,dx \leqslant \Big(\int_a^b \big|f(x)\big|^2\,dx\Big)^{\frac{1}{2}}\Big(\int_a^b \big|g(x)\big|^2\,dx\Big)^{\frac{1}{2}}\,.$$