Exercise Problem Sets 4

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Problem 1. Let f be a function defined on an open interval containing c. Show that f is differentiable at c if and only if there exists a real number L satisfying that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\big|f(c+h) - f(c) - Lh\big| \leqslant \varepsilon|h| \qquad \text{whenever} \quad |h| < \delta\,.$$

Hint: See the (first part of the) proof of the chain rule for reference.

Problem 2. Let f, g be functions defined on an open interval, and $n \in \mathbb{N}$. Show that if the n-th derivatives of f and g exist on I, then

$$\begin{aligned}\frac{d^n}{dx^n}(fg)(x) &= f^{(n)}(x)g(x) + C^n_1 f^{(n-1)}(x)g'(x) + C^n_2 g^{(n-2)}(x)g''(x) + \cdots \\ &\quad + C^n_{n-2}f''(x)g^{(n-2)}(x) + C^n_{n-1}f'(x)g^{(n-1)}(x) + f(x)g^{(n)}(x) \\ &= \sum_{k=0}^{n} C^n_k f^{(n-k)}(x)g^{(k)}(x)\,,\end{aligned}$$

where $C^n_k = \dfrac{n!}{k!(n-k)!}$ is "n choose k".

Hint: Prove by induction.

Problem 3. Let I be an open interval and $c \in I$. The left-hand and right-hand derivative of f at c, denoted by $f'(c^+)$ and $f'(c^-)$, respectively, are defined by

$$f'(c^+) = \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h} \qquad \text{and} \qquad f'(c^-) = \lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$$

provides the limits exist.

1. Show that if f is differentiable at c if and only if $f'(c^+) = f'(c^-)$, and in either case we have $f'(c) = f'(c^+) = f'(c^-)$.
2. Let $f(x) = \begin{cases} x^2 & \text{if } x \leqslant 2\,, \\ mx + k & \text{if } x > 2\,. \end{cases}$ Find the value of m and k such that f is differentiable at 2.
3. Is there a value of b that will make

$$g(x) = \begin{cases} x + b & \text{if } x < 0\,, \\ \cos x & \text{if } x \geqslant 0\,. \end{cases}$$

continuous at 0? Differentiable at 0? Give reasons for your answers.

Problem 4. 1. Let $n \in \mathbb{N}$. Show that $\sum\limits_{k=1}^{n-1} kx^{k-1} = \dfrac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}$ if $x \neq 1$.

2. Show that $\sum\limits_{k=1}^{n} k\cos(kx) = \dfrac{-1 + (2n+1)\sin\frac{x}{2}\sin(n+\frac{1}{2})x + \cos\frac{x}{2}\cos(n+\frac{1}{2})x}{4\sin^2\frac{x}{2}}$ if $x \in (-\pi, \pi)$.

Hint 1. Find the sum $\sum\limits_{k=1}^{n-1} x^k$ first and then observe that $\sum\limits_{k=1}^{n-1} kx^{k-1} = \sum\limits_{k=1}^{n-1} \dfrac{d}{dx}x^k$.

2. Find the sum $\sum_{k=1}^{n} \sin(kx)$ first and then observe that $\sum_{k=1}^{n} k\cos(kx) = \sum_{k=1}^{n} \frac{d}{dx}\sin(kx)$.

Problem 5. For a fixed constant $a > 1$, consider the function $f(x) = \log_a x$. Suppose that you are given the fact that the limit

$$\lim_{h\to 0} \frac{\log_{10}(1+h)}{h} \approx 0.43429$$

exists.

1. Show that f is differentiable on $(0,\infty)$ for all $a > 1$.

2. Show that there exists $a > 1$ such that $f'(x) = \frac{1}{x}$ for all $x \in (0,\infty)$.

Hint: 1. Use the "change of base formula"（換底公式）for logarithm.

2. Define $g(a) = \left.\frac{d}{dx}\right|_{x=1} \log_a x$. Apply the intermediate value theorem to g.

Problem 6. Let $f(x) = a_1 \sin x + a_2 \sin(2x) + a_3 \sin(3x) + \cdots + a_n \sin(nx)$, where $a_1, a_2, \cdots, a_n$ are real numbers and $n \in \mathbb{N}$. Show that if $|f(x)| \leqslant |\sin x|$ for all $x \in \mathbb{R}$, then

$$|a_1 + 2a_2 + 3a_3 + \cdots + na_n| \leqslant 1\,.$$

Problem 7. Let $k \in \mathbb{N}$. Suppose that $\frac{d^n}{dx^n}\frac{1}{x^k-1} = \frac{p_n(x)}{(x^k-1)^{n+1}}$. Find the degree of p_n and $p_n(1)$.

Problem 8. Let $f_1, f_2, \cdots, f_n : \mathbb{R} \to \mathbb{R}$ be differentiable functions (that is, f_j is differentiable on $\mathbb{R}$ for all $1 \leqslant j \leqslant n$), and

$$h(x) = (f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1)(x)\,.$$

Show that

$$h'(x) = f_n'\big(g_{n-1}(x)\big) \cdot f_{n-2}'\big(g_{n-2}(x)\big) \cdot \cdots \cdot f_2'\big(g_1(x)\big) \cdot f_1'(x)\,.$$

where $g_k = f_k \circ f_{k-1} \circ \cdots \circ f_2 \circ f_1$.

Hint: Prove by induction.

Problem 9. 1. Let $r \in \mathbb{Q}$, and $f : (0,\infty) \to \mathbb{R}$ be defined by $f(x) = x^r$. Find the derivative of f.

2. Find the derivatives of $y = x^{\frac{1}{4}}$ and $y = x^{\frac{3}{4}}$ by the fact that $x^{\frac{1}{4}} = \sqrt{\sqrt{x}}$ and $x^{\frac{3}{4}} = \sqrt{x\sqrt{x}}$.

3. Let $g : (a,b) \to \mathbb{R}$ be differentiable. Find the derivative of $y = |g(x)|$.

Problem 10. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and satisfy $f\big(\frac{x^2-1}{x^2+1}\big) = x$ for all $x > 0$. Find $f'(0)$.

Problem 11. 1. Let $n \in \mathbb{N}$. Show that $\frac{d}{dx}\big[\sin^n x \cos(nx)\big] = n\sin^{n-1} x \cos(n+1)x\,.$

2. Find a similar formula for the derivative of $\cos^n x \cos(nx)$.

Problem 12. Find the derivative of the following functions:

1. $y = \cos\sqrt{\sin(\tan(\pi x))}$.

2. $y = \big[x + (x + \sin^2 x)^3\big]^4$.