Exercise Problem Sets 1

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Problem 1. Let θ be a real number such that $t = \tan\frac{\theta}{2}$ also be a real number. Show that

$$\sin\theta = \frac{2t}{1+t^2} \qquad \text{and} \qquad \cos\theta = \frac{1-t^2}{1+t^2}\,.$$

Problem 2. （這題雖不指定上臺講解，但請所有同學都能好好閱讀證明，了解整個流程為什麼就證明了 $\lim\limits_{x\to c} f(x) = 0$）Let $f : (0,\infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{p} & \text{if } x = \dfrac{q}{p},\ p, q \in \mathbb{N} \text{ and } (p,q) = 1\,, \\ 0 & \text{if } x \text{ is irrational}\,. \end{cases}$$

Proof. Let $\varepsilon > 0$ be given. Then there exists a prime number p such that $\frac{1}{p} < \varepsilon$. Let $q_1, q_2, \cdots, q_n$ be rational numbers in $(\frac{c}{2}, \frac{3c}{2})$ satisfying

$$q_j = \frac{s}{r}\,, (r,s) = 1, 1 \leqslant r \leqslant p\,,$$

and define $\delta = \frac{1}{2}\min\Big(\{|c-q_1|, |c-q_2|, \cdots, |c-q_n|\}\backslash\{0\}\Big)$. Then $\delta > 0$. Suppose that x satisfies that $0 < |x-c| < \delta$.

1. If $x \in \mathbb{Q}^\complement$, then $f(x) = 0$ which shows that $|f(x)| < \varepsilon$.
2. If $x \in \mathbb{Q}$, then $x = \frac{s}{r}$ for some natural numbers r, s satisfying $(r,s) = 1$. By the choice of δ, we find that $r > p$; thus
$$\big|f(x)\big| = \frac{1}{r} < \frac{1}{p} < \varepsilon\,.$$

In either case, $\big|f(x)\big| < \varepsilon$; thus we establish that

$$\big|f(x) - 0\big| < \varepsilon \quad \text{whenever} \quad 0 < |x-c| < \delta\,.$$

Therefore, $\lim\limits_{x\to c} f(x) = 0$. □

Problem 3. Let f be given in Problem 2 and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} f(x) & \text{if } x > 0\,, \\ f(-x) & \text{if } x < 0\,, \\ 1 & \text{if } x = 0\,. \end{cases}$$

Find $\lim\limits_{x\to 0} g(x)$.