Exercise Problem Sets 12

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Problem 1. Evaluate the following limits. Use L'Hôspital's Rule where appropriate. If L'Hôspital's Rule does not apply, explain why.

1. $\lim_{x\to 0^+} \dfrac{\arctan(2x)}{\ln x}$.   2. $\lim_{x\to 0^+} \dfrac{x^x-1}{\ln x + x - 1}$.   3. $\lim_{x\to 0} \dfrac{\ln(1+x)}{\cos x + e^x - 1}$.

4. $\lim_{x\to 0} \dfrac{x^a-1}{x^b-1}$, where $b \neq 0$.   5. $\lim_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$.   6. $\lim_{x\to a^+} \dfrac{\cos x \cdot \ln(x-a)}{\ln(e^x - e^a)}$.

7. $\lim_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\arctan x}\right)$.   8. $\lim_{x\to\infty} (x - \ln x)$.   9. $\lim_{x\to 1^+} \ln(x^7-1) - \ln(x^5-1)$.

10. $\lim_{x\to\infty} x^{\frac{\ln 2}{1+\ln x}}$.   11. $\lim_{x\to\infty} x^{e^{-x}}$.   12. $\lim_{x\to 1} (2-x)^{\tan(\pi x/2)}$.   13. $\lim_{x\to 0^+} (\sin x)(\ln x)$.

Problem 2. Evaluate the following limits:

1. $\lim_{x\to\infty} x\Big[\big(1+\dfrac{1}{x}\big)^x - e\Big]$.   2. $\lim_{x\to\infty} \Big\{\dfrac{e}{2}x + x^2\Big[\big(1+\dfrac{1}{x}\big)^x - e\Big]\Big\}$.

3. $\lim_{x\to\infty} x\Big[\big(1+\dfrac{1}{x}\big)^x - e\ln\big(1+\dfrac{1}{x}\big)^x\Big]$.   4. $\lim_{x\to 0} \big(\dfrac{\sin x}{x}\big)^{\frac{1}{x^2}}$.   5. $\lim_{x\to\infty} \big(\dfrac{\sin x}{x}\big)^{\frac{1}{x^2}}$.

6. $\lim_{x\to\infty} \Big(x - x^2 \ln\dfrac{1+x}{x}\Big)$.   7. $\lim_{x\to\infty} \Big[\dfrac{1}{x}\cdot\dfrac{a^x-1}{a-1}\Big]^{\frac{1}{x}}$, where $a > 0$ and $a \neq 1$.

Problem 3. For what values of a and b is the following equations true?

1. $\lim_{x\to 0} \left(\dfrac{\sin 2x}{x^3} + a + \dfrac{b}{x^2}\right) = 0$.

2. $\lim_{x\to 0} \left(\dfrac{\tan 2x}{x^3} + \dfrac{a}{x^2} + \dfrac{\sin bx}{x}\right) = 0$.

Problem 4. Show that $\lim_{x\to\infty} x^{x^{-n}} = 1$ for every positive integer n.

Problem 5. Let $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$

1. Find $f'(0)$. Is f continuously differentiable?

2. Show that f has derivatives of all orders on $\mathbb{R}$; that is, f is infinitely many times differentiable on $\mathbb{R}$.

Hint: First show by induction that there is a polynomial $p_n(x)$ and a non-negative integer k_n such that $f^{(n)}(x) = \dfrac{p_n(x)f(x)}{x^{k_n}}$ for $x \neq 0$.

Problem 6. Find $\dfrac{d}{dx}\arcsin(\sin x)$, $\dfrac{d}{dx}\arccos(\sin x)$ and $\dfrac{d}{dx}\arctan(\tan x)$.

Problem 7. Show that $2\arcsin x = \arccos(1-2x^2)$ for all $x \geqslant 0$.

Problem 8. Prove the identity $\arcsin \frac{x-1}{x+1} = 2\arctan\sqrt{x} - \frac{\pi}{2}$ for all $x \geqslant 0$.

Problem 9. Prove that $\frac{x}{1+x^2} < \arctan x < x$ for all $x > 0$.

Problem 10. Evaluate $\int_0^1 \arcsin x \, dx$ by interpreting it as an area and integrating with respect to y instead of x.

Problem 11. Evaluate the following definite integrals.

1. $\int_0^{\frac{1}{\sqrt{2}}} \frac{\arcsin x}{\sqrt{1-x^2}} \, dx$.
2. $\int_0^{\frac{1}{\sqrt{2}}} \frac{\arccos x}{\sqrt{1-x^2}} \, dx$.
3. $\int_{\ln 2}^{\ln 4} \frac{e^{-x}}{\sqrt{1-e^{-2x}}} \, dx$.
4. $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}} \, dx$.
5. $\int_3^4 \frac{dx}{(x-1)\sqrt{x^2-2x}}$.

Problem 12. Find the following indefinite integrals.

1. $\int \sqrt{e^x - 3} \, dx$.
2. $\int \frac{\sqrt{x-2}}{x+1} \, dx$.
3. $\int \frac{dx}{\sqrt{-2x^2+8x+4}}$.
4. $\int \frac{2x \arctan(x^2+1)}{x^4+2x^2+2} \, dx$.
5. $\int \frac{\sqrt{x}}{4+x^3} \, dx$.
6. $\int \sqrt{\frac{x}{4+x^3}} \, dx$, $x > 0$.

Problem 13. Find the function y satisfying $(1+x^2)y' + xy = 1$ and $y(0) = 1$.