Exercise Problem Sets 11

Nov. 29. 2019

Problem 1. Show that the following functions are decreasing on $(0,\infty)$.

1. $y = \big(1 + \dfrac{1}{2x}\big)^{x+0.5}$. 2. $y = \big(1 + \dfrac{1}{x}\big)^{x+0.5}$.

Problem 2. In this example you are asked to compute the integral of $y = xe^x$ by the Riemann sum. Complete the following.

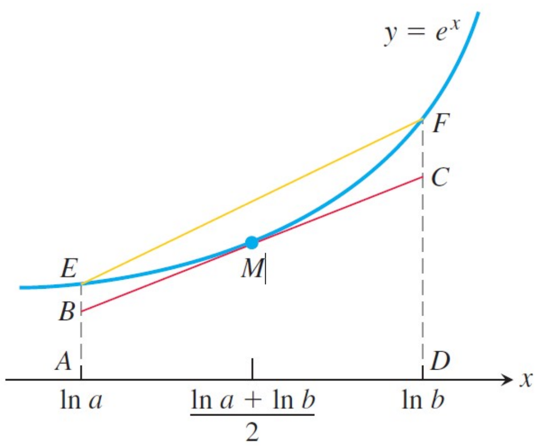
1. Show that if $r \neq 1$, then $\sum\limits_{k=1}^{n} kr^k = \dfrac{r(1-r^n)}{(1-r)^2} - \dfrac{nr^{n+1}}{1-r}$.
2. Compute $\displaystyle\int_0^a xe^x\,dx$ by the limit the Riemann sum of $y = xe^x$ for regular partition using the right end-point rule.
3. Find an anti-derivative of $y = xe^x$.

Problem 3 (Integrating Factor).

1. Let $f, g : [a,b] \to \mathbb{R}$ be a continuous function, F be an anti-derivative of f, and $y : [a,b] \to \mathbb{R}$ satisfies that
$$y' + f(x)y = g(x)\,. \qquad (\star)$$
Find an expression of y.
2. Find the function y satisfying $y' + x^2 y = 2x^3$ and $y(0) = 1$.

Hint: Multiply both sides of $(\star)$ by $\exp(F(x))$ and observe that the left-hand side is the derivative of a certain function.

Problem 4. 1. Show that for $0 < a < b$,
$$e^{\frac{\ln a + \ln b}{2}} \cdot (\ln b - \ln a) < \int_{\ln a}^{\ln b} e^x\,dx < \frac{e^{\ln a} + e^{\ln b}}{2} \cdot (\ln b - \ln a)\,.$$



2. Using the result above to show that for $0 < a < b$,

$$\sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2}.$$

Problem 5. Prove the following inequalities.

1. $e^x > 1 + \ln(1+x)$ for all $x > 0$.

2. $e^x > 1 + (1+x)\ln(1+x)$ for all $x > 0$.

3. $e^x \geqslant 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots + \dfrac{x^n}{n!}$ for all $x \geqslant 0$ and $n \in \mathbb{N}$.

Problem 6. Let a, b be two positive numbers, p, q any nonzero numbers, and $p < q$. Prove that

$$\left[\theta a^p + (1-\theta)b^p\right]^{\frac{1}{p}} \leqslant \left[\theta a^q + (1-\theta)b^q\right]^{\frac{1}{q}} \qquad \forall\, \theta \in (0,1).$$

Hint: Show that the function $f(p) = \left[\theta a^p + (1-\theta)b^p\right]^{\frac{1}{p}}$ is an increasing function of p.

Problem 7. 1. Find an equation for the line through the origin tangent to the graph of $y = \ln x$.

2. Show that $\ln x < \dfrac{x}{e}$ for all $x \neq e$.

3. Show that $x^e < e^x$ for all $x \neq e$.

4. Show that if $e \leqslant A < B$, then $A^B > B^A$.

Problem 8 (Implicit Differentiation).

1. Find y' if $e^{\frac{x}{y}} = x - y$.

2. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

3. Find an equation of the tangent line to the curve $1 + \ln xy = e^{x-y}$ at the point $(1, 1)$