微積分 MA1002-A 上課筆記（精簡版）

2019.06.11.

Ching-hsiao Arthur Cheng 鄭經斅

14.5 Change of Variables Formula

In this section, we consider the version of substitution of variables in multiple integrals. We have used the technique of substitution of variable to evaluate the iterated integrals in, for example, Example 14.12 and 14.13; however, these substitutions of variable always assume that other variables are independent of the new variable introduced by the substitution of variable. We would like to investigate the effect of making a change of variables such as $x = r\cos\theta$, $y = r\sin\theta$ in computing the double integrals.

14.5.1 Double integrals in polar coordinates

We start our discussion with double integrals in polar coordinates. Suppose that R is the shaded region shown in Figure 14.4 and $f : R \to \mathbb{R}$ is continuous.

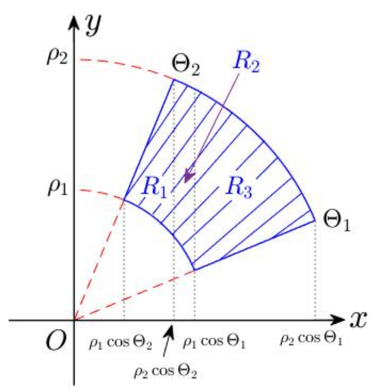

Figure 14.3: Rectangle in polar coordinates

Then to compute the double integral $\displaystyle\iint\limits_R f(x,y)\,dA$ using the Fubini theorem directly, we need to divide R into three sub-regions R_1, R_2, R_3 given by

$$R_1 = \left\{(x,y) \,\middle|\, \rho_1\cos\Theta_2 \leqslant x \leqslant \rho_2\cos\Theta_2, \sqrt{\rho_1^2 - x^2} \leqslant y \leqslant x\tan\Theta_2\right\},$$
$$R_2 = \left\{(x,y) \,\middle|\, \rho_2\cos\Theta_2 \leqslant x \leqslant \rho_1\cos\Theta_1, \sqrt{\rho_2^2 - x^2} \leqslant y \leqslant \sqrt{\rho_1^2 - x^2}\right\},$$
$$R_3 = \left\{(x,y) \,\middle|\, \rho_1\cos\Theta_1 \leqslant x \leqslant \rho_2\Theta_2, x\tan\Theta_1 \leqslant y \leqslant \sqrt{\rho_2^2 - x^2}\right\},$$

and write

$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R_1} f(x,y)\,dA + \iint\limits_{R_2} f(x,y)\,dA + \iint\limits_{R_3} f(x,y)\,dA\,.$$

However, we know that the region R above is a rectangle in $r\theta$-plane, where (r,θ) is the polar coordinates on the plane. To be more precise, in polar coordinate the region R can be expressed as $R' \equiv \{(r,\theta) \,|\, \rho_1 \leqslant r \leqslant \rho_2, \Theta_1 \leqslant \theta \leqslant \Theta_2\}$, which means that every point (x,y) in R can be written as $(r\cos\theta, r\sin\theta)$ for $(r,\theta) \in R'$, and vice versa. One should expect that it should be easier to write down the iterated integral for computing $\displaystyle\iint_R f(x,y)\,dA$.

Let $\mathcal{P}_r = \{\rho_1 = r_0 < r_1 < \cdots < r_n = \rho_2\}$ and $\mathcal{P}_\theta = \{\Theta_1 = \theta_0 < \theta_1 < \cdots < \theta_m = \Theta_2\}$ be partitions of $[\rho_1,\rho_2]$ and $[\Theta_1,\Theta_2]$, respectively, $R_{ij} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j]$ be rectangles in the $r\theta$-plane, S_{ij} be the sub-region in the xy-plane corresponds to R_{ij} under the polar coordinate; that is,

$$S_{ij} = \{(r\cos\theta, r\sin\theta) \,|\, r \in [r_{i-1}, r_i], \theta \in [\theta_{j-1}, \theta_j]\}\,.$$

The collection $\mathcal{P} = \{S_{ij} \,|\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$ is called a partition of rectangles in polar coordinates, and the norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is the maximum diameter of S_{ij}.

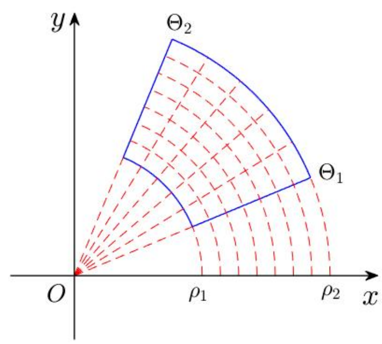

Figure 14.4: Rectangle in polar coordinates

A Riemann sum of f for partition $\mathcal{P}$ is of the form $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} f(\xi_{ij},\eta_{ij})|S_{ij}|$, where $|S_{ij}|$ is the area of S_{ij} and $\{(\xi_{ij},\eta_{ij})\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ be collection of points satisfying $(\xi_{ij},\eta_{ij}) \in S_{ij}$. Then intuitively $\displaystyle\iint_R f(x,y)\,dA$ is the limit of Riemann sums of f for $\mathcal{P}$ as $\|\mathcal{P}\|$ approaches zero.

To see the limit of Riemann sums, we choose a particular partition $\mathcal{P}$ and collection $\{(\xi_{ij},\eta_{ij})\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$. We equally partition $[\rho_1,\rho_2]$ and $[\Theta_1,\Theta_2]$ into n and m sub-intervals. Let $\Delta r = \dfrac{\rho_2-\rho_1}{n}$ and $\Delta\theta = \dfrac{\Theta_2-\Theta_1}{m}$, and $r_i = \rho_1 + i\Delta r$ and $\theta_j = \Theta_1 + j\Delta\theta$, and $\xi_{ij} = r_i\cos\theta_j$ and $\eta_{ij} = r_i\sin\theta_j$. Noting that

$$|S_{ij}| = \frac{1}{2}(r_i^2 - r_{i-1}^2)(\theta_j - \theta_{j-1}) = \frac{1}{2}(r_i + r_{i-1})\Delta r\Delta\theta = r_i\Delta r\Delta\theta - \frac{1}{2}\Delta r^2\Delta\theta\,,$$

we find that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(\xi_{ij},\eta_{ij})|S_{ij}| = \sum_{i=1}^{n}\sum_{j=1}^{m} f(r_i\cos\theta_j, r_i\sin\theta_j)r_i\Delta r\Delta\theta - \frac{\Delta r}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} f(r_i\cos\theta_j, r_i\sin\theta_j)\Delta r\Delta\theta\,.$$

Let $g(r,\theta) = rf(r\cos\theta, r\sin\theta)$ and $h(r,\theta) = f(r\cos\theta, r\sin\theta)$, then

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(\xi_{ij},\eta_{ij})|S_{ij}| = \sum_{i=1}^{n}\sum_{j=1}^{m} g(r_i,\theta_j)\Delta r\Delta\theta - \frac{\Delta r}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} h(r_i,\theta_j)\Delta r\Delta\theta\,.$$

As n,m approach ∞, we find that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} g(r_i,\theta_j)\Delta r\Delta\theta \to \iint_{R'} g(r,\theta)\,d(r,\theta) = \iint_{R'} f(r\cos\theta, r\sin\theta)r\,d(r,\theta)\,,$$
$$\sum_{i=1}^{n}\sum_{j=1}^{m} h(r_i,\theta_j)\Delta r\Delta\theta \to \iint_{R'} h(r,\theta)\,d(r,\theta) = \iint_{R'} f(r\cos\theta, r\sin\theta)\,d(r,\theta)\,,$$

where the right-hand side integrals denotes the double integrals on the rectangle R'. Therefore, the limit of Riemann sums of f for $\mathcal{P}$ as $\|\mathcal{P}\|$ approaches zero is

$$\iint_{R'} f(r\cos\theta, r\sin\theta)r\,d(r,\theta)\,;$$

thus

$$\iint_{R} f(x,y)\,d(x,y) = \iint_{R'} f(r\cos\theta, r\sin\theta)r\,d(r,\theta)\,. \tag{14.5.1}$$

14.5.2 Jacobian

Recall the substitution of variables formula for the integral of functions of one variable:

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du\,.$$

Suppose that $g:[a,b]\to\mathbb{R}$ is one-to-one. If g is increasing, then $g'\geqslant 0$ and $g([a,b]) = \big[g(a),g(b)\big]$; thus the formula above can be rewritten as

$$\int_{g([a,b])} f(u)\,du = \int_{[a,b]} f(g(x))g'(x)\,dx = \int_{[a,b]} f(g(x))\big|g'(x)\big|\,dx\,.$$

If g is decreasing, then $g' \leqslant 0$ and $g([a,b]) = \big[g(b), g(a)\big]$; thus the formula above can be written as

$$\int_{g([a,b])} f(u)\,du = -\int_{[a,b]} f(g(x))g'(x)\,dx = \int_{[a,b]} f(g(x))\big|g'(x)\big|\,dx\,.$$

Therefore, in either cases we have a rewritten version of the substitution of variable formula

$$\int_{g([a,b])} f(u)\,du = \int_{[a,b]} f(g(x))\big|g'(x)\big|\,dx\,.$$

In this section, we are concerned with the substitution of variable formula (usually called the change of variables formula in the case of multiple integrals) for double and triple integrals, here the substitution of variables is usually given by $x = x(u,v), y = y(u,v)$ for the case of double integrals and $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ for the case of triple integrals.

Consider the double integral $\displaystyle\iint\limits_R f(x,y)\,dA$. Suppose that we have the change of variables $x = x(u,v)$ and $y = y(u,v)$, and the Fubini Theorem implies that the double integral can be written as $\displaystyle\int\Big(\int f(x,y)\,dy\Big)dx$, here we do not write the upper limit and lower limit explicitly. Note the inner integral in the iterated integral is computed by assuming that y is fixed. When x is a fixed constant, the relation $x = x(u,v)$ gives a relation between u and v, and the implicit differentiation provides that

$$\frac{du}{dv} = -\frac{x_v(u,v)}{x_u(u,v)}$$

if $x_u \neq 0$. Making the substitution of the variable $y = y(u,v)$ with u, v satisfying the relation $x = x(u,v)$, we find that

$$\begin{aligned} dy &= y_u(u,v)du + y_v(u,v)dv = y_u(u,v)\frac{du}{dv}dv + y_v(u,v)dv \\ &= \frac{x_u(u,v)y_v(u,v) - x_v(u,v)y_u(u,v)}{x_u(u,v)}\,dv\,; \end{aligned}$$

thus

$$\int f(x,y)\,dy = \int f(x(u,v),y(u,v))\Big|\frac{x_u(u,v)y_v(u,v) - x_v(u,v)y_u(u,v)}{x_u(u,v)}\Big|\,dv\,.$$

Therefore, the substitution of variable $x = x(u,v)$, where "v is treated as a constant since it has been integrated", is

$$\begin{aligned}
&\int \Big(\int f(x,y)\,dy \Big) dx \\
&\quad = \int \Big(\int f(x(u,v),y(u,v)) \Big| \frac{x_u(u,v)y_v(u,v) - x_v(u,v)y_u(u,v)}{x_u(u,v)} \Big|\, dv \Big) |x_u(u,v)|\, du \\
&\quad = \int \Big(\int f(x(u,v),y(u,v)) \big| x_u(u,v)y_v(u,v) - x_v(u,v)y_u(u,v) \big|\, dv \Big) du\,. \qquad (14.5.2)
\end{aligned}$$

Example 14.19. Consider the change of variables using polar coordinate $x = r\cos\theta$, $y = r\sin\theta$ (treat r, θ as the u, v variables, respectively). Then

$$|x_u y_v - x_v y_u| = |\cos\theta \cdot r\cos\theta - (-r\sin\theta)\cdot\sin\theta| = |r| = r\,;$$

thus (14.5.2) implies the change of variables formula for polar coordinates (14.5.1).

Now we consider the possible change of variables formula for triple integrals. Suppose that by the Fubini Theorem,

$$\iiint_Q f(x,y,z)\,dV = \int \Big[\int \Big(\int f(x,y,z)\,dz \Big) dy \Big] dx\,,$$

where again we do not state explicitly the upper and the lower limit of each integral. For a given change of variables $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$, the first integral that we need to evaluate is $\int f(x,y,z)\,dz$, and this integral is computed by assuming that x, y are fixed constants. When x and y are fixed constants, the relations $x = x(u,v,w)$ and $y = y(u,v,w)$ give relations among u, v, w. Suppose that these relations imply that u and v are differentiable functions of w, then the implicit differentiation (when applicable) provides that

$$\begin{aligned}
0 &= x_u(u,v,w)\frac{du}{dw} + x_v(u,v,w)\frac{dv}{dw} + x_w(u,v,w)\,, \\
0 &= y_u(u,v,w)\frac{du}{dw} + y_v(u,v,w)\frac{dv}{dw} + y_w(u,v,w)\,;
\end{aligned}$$

thus if $x_u y_v - x_v y_u \neq 0$, we have

$$\begin{aligned}
\frac{du}{dw} &= \frac{x_v(u,v,w)y_w(u,v,w) - x_w(u,v,w)y_v(u,v,w)}{x_u(u,v,w)y_v(u,v,w) - x_v(u,v,w)y_u(u,v,w)}\,, \\
\frac{dv}{dw} &= \frac{x_w(u,v,w)y_u(u,v,w) - x_u(u,v,w)y_w(u,v,w)}{x_u(u,v,w)y_v(u,v,w) - x_v(u,v,w)y_u(u,v,w)}\,,
\end{aligned}$$

and these identities further imply that

$$\begin{aligned}
dz &= z_u(u,v,w)du + z_v(u,v,w)dv + z_w(u,v,w)dw \\
&= \Big[z_u \frac{x_v y_w - x_w y_v}{x_u y_v - x_v y_u} + z_v \frac{x_w y_u - x_u y_w}{x_u y_v - x_v y_u} + z_w\Big](u,v,w)dw \\
&= \Big[\frac{x_v y_w z_u - x_w y_v z_u + x_w y_u z_v - x_u y_w z_v + x_u y_v z_w - x_v y_u z_w}{x_u y_v - x_v y_u}\Big](u,v,w)dw\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int f(x,y,z)\,dz = \int & f(x(u,v,w), y(u,v,w), z(u,v,w))\times \\
& \times \Big|\frac{x_v y_w z_u - x_w y_v z_u + x_w y_u z_v - x_u y_w z_v + x_u y_v z_w - x_v y_u z_w}{x_u y_v - x_v y_u}\Big|(u,v,w)\,dw\,,
\end{aligned}$$

and (14.5.2), by treating w as a constant since it has been integrated, implies that

$$\begin{aligned}
&\int \Big[\int \Big(\int f(x,y,z)\,dz\Big)dy\Big]dx \\
&= \int \Big[\int \Big(\int f(x(u,v,w), y(u,v,w), z(u,v,w))\times \\
&\qquad \times \Big|\frac{x_v y_w z_u - x_w y_v z_u + x_w y_u z_v - x_u y_w z_v + x_u y_v z_w - x_v y_u z_w}{x_u y_v - x_v y_u}\Big|(u,v,w)\,dw\Big)\times \\
&\qquad \times \big|x_u(u,v,w)y_v(u,v,w) - x_v(u,v,w)y_u(u,v,w)\big|\,dv\Big]du \\
&= \int \Big[\int \Big(\int f(x(u,v,w), y(u,v,w), z(u,v,w))\times \\
&\qquad \times \big|x_v y_w z_u - x_w y_v z_u + x_w y_u z_v - x_u y_w z_v + x_u y_v z_w - x_v y_u z_w\big|(u,v,w)\,dw\Big)dv\Big]du\,.
\end{aligned}$$

The naive (but **wrong**) computations above motivate the following

Definition 14.20

If $x = x(u,v)$ and $y = y(u,v)$, the ***Jacobian*** of x and y with respect to u and v, denoted by $\dfrac{\partial(x,y)}{\partial(u,v)}$, is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u\,.$$

If $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$, the ***Jacobian*** of x, y and z with respect to u, v and w, denoted by $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$, is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = x_u y_v z_w + x_w y_u z_v + x_v y_w z_u - x_w y_v z_u - x_v y_u z_w - x_u y_w z_v\,.$$