微積分 MA1002-A 上課筆記（精簡版）

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Theorem 14.11

Let R be a closed region in the plane, and $f : R \to \mathbb{R}$ be a continuously differentiable function. Then the area of the surface $S = \{(x,y,z) \,|\, (x,y) \in R\,, z = f(x,y)\}$ is given by

$$\iint\limits_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dA\,.$$

Example 14.13. Find the surface area of the paraboloid $z = 1 + x^2 + y^2$ that lies above the unit disk.

Let $f(x,y) = 1+x^2+y^2$ and $R = \{(x,y) \,|\, x^2+y^2 \leqslant 1\}$. Then the surface area of interest is given by

$$\iint\limits_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dA = \iint\limits_R \sqrt{1+4x^2+4y^2}\, dA\,.$$

Since R can also be expressed as $R = \{(x,y) \,|\, -1 \leqslant x \leqslant 1, -\sqrt{1-x^2} \leqslant y \leqslant \sqrt{1-x^2}\}$, the Fubini Theorem then implies that

$$\iint\limits_R \sqrt{1+4x^2+4y^2}\, dA = \int_{-1}^{1} \Big(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1+4x^2+4y^2}\, dy \Big) dx\,.$$

By the fact that $\displaystyle\int \sqrt{a^2+b^2u^2}\, du = \frac{a^2}{2b}\Big[\frac{bu\sqrt{a^2+b^2u^2}}{a^2} + \ln\big(bu+\sqrt{a^2+b^2u^2}\big)\Big] + C$ if $a, b > 0$, we find that

$$\begin{aligned}
\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1+4x^2+4y^2}\, dy &= 2\int_0^{\sqrt{1-x^2}} \sqrt{1+4x^2+4y^2}\, dy \\
&= \frac{1+4x^2}{2}\Big[\frac{2y\sqrt{1+4x^2+4y^2}}{1+4x^2} + \ln\big(2y+\sqrt{1+4x^2+4y^2}\big)\Big]\Big|_{y=0}^{y=\sqrt{1-x^2}} \\
&= \sqrt{5}\sqrt{1-x^2} + \frac{1+4x^2}{2}\ln\frac{\sqrt{5}+2\sqrt{1-x^2}}{\sqrt{1+4x^2}}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\iint\limits_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dA &= \int_{-1}^{1} \Big[\sqrt{5}\sqrt{1-x^2} + \frac{1+4x^2}{2}\ln\frac{\sqrt{5}+2\sqrt{1-x^2}}{\sqrt{1+4x^2}}\Big] dx \\
&= \frac{\sqrt{5}}{2}\pi + \frac{1}{2}\int_{-1}^{1} (1+4x^2)\ln\frac{\sqrt{5}+2\sqrt{1-x^2}}{\sqrt{1+4x^2}}\, dx\,.
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}
&\int_{-1}^{1} (1+4x^2) \ln \frac{\sqrt{5}+2\sqrt{1-x^2}}{\sqrt{1+4x^2}}\, dx \\
&\quad = \Big(x+\frac{4}{3}x^3\Big) \ln \frac{\sqrt{5}+2\sqrt{1-x^2}}{\sqrt{1+4x^2}}\Big|_{x=-1}^{x=1} - \int_{-1}^{1} \Big(x+\frac{4}{3}x^3\Big) \frac{d}{dx} \ln \frac{\sqrt{5}+2\sqrt{1-x^2}}{\sqrt{1+4x^2}}\, dx \\
&\quad = -\int_{-1}^{1} \Big(x+\frac{4}{3}x^3\Big) \frac{\sqrt{1+4x^2}}{\sqrt{5}+2\sqrt{1-x^2}} \frac{\dfrac{-2x}{\sqrt{1-x^2}}\sqrt{1+4x^2} - \dfrac{4x}{\sqrt{1+4x^2}}(\sqrt{5}+2\sqrt{1-x^2})}{1+4x^2}\, dx \\
&\quad = -\int_{-1}^{1} \Big(x+\frac{4}{3}x^3\Big) \frac{-2x}{\sqrt{5}+2\sqrt{1-x^2}} \frac{5+2\sqrt{5}\sqrt{1-x^2}}{(1+4x^2)\sqrt{1-x^2}}\, dx \\
&\quad = \frac{\sqrt{5}}{3} \int_{-1}^{1} \frac{2x(3x+4x^3)}{(1+4x^2)\sqrt{1-x^2}}\, dx = \frac{\sqrt{5}}{3} \int_{-1}^{1} \frac{-1+3(1+4x^2)-2(1-x^2)(1+4x^2)}{(1+4x^2)\sqrt{1-x^2}}\, dx \\
&\quad = \frac{-\sqrt{5}}{3} \int_{-1}^{1} \frac{1}{(1+4x^2)\sqrt{1-x^2}}\, dx + \sqrt{5} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx - \frac{2\sqrt{5}}{3} \int_{-1}^{1} \sqrt{1-x^2}\, dx \\
&\quad = \frac{-\sqrt{5}}{3} \int_{-1}^{1} \frac{1}{(1+4x^2)\sqrt{1-x^2}}\, dx + \frac{2\sqrt{5}}{3}\pi\,.
\end{aligned}$$

By the substitution of variable $x = \sin\theta$, we find that

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{(1+4x^2)\sqrt{1-x^2}}\, dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{1+4\sin^2\theta}\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{1+2(1-\cos 2\theta)}\, d\theta \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3-2\cos 2\theta}\, d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1}{3-2\cos\phi}\, d\phi\,.
\end{aligned}$$

By the substitution of variable $\tan\frac{\phi}{2} = t$, we further obtain that

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{(1+4x^2)\sqrt{1-x^2}}\, dx &= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{3-2\frac{1-t^2}{1+t^2}} \frac{2dt}{1+t^2} = \int_{-\infty}^{\infty} \frac{1}{1+5t^2}\, dt \\
&= \frac{1}{\sqrt{5}} \arctan(\sqrt{5}t)\Big|_{t=-\infty}^{t=\infty} = \frac{\pi}{\sqrt{5}}\,.
\end{aligned}$$

Therefore,

$$\iint\limits_R \sqrt{1+f_x(x,y)^2+f_y(x,y)^2}\, dA = \frac{\sqrt{5}}{2}\pi + \frac{1}{2}\Big[-\frac{\sqrt{5}}{3}\cdot\frac{\pi}{\sqrt{5}} + \frac{2\sqrt{5}\pi}{3}\Big] = \frac{\pi}{6}(5\sqrt{5}-1)\,.$$

14.4 Triple Integrals and Applications

Let Q be a bounded region in the space, and $f : Q \to \mathbb{R}$ be a non-negative function which described the point density of the region. We are interested in the mass of Q.

We start with the simple case that $Q = [a, b] \times [c, d] \times [r, s]$ is a cube. Let

$$\begin{aligned}\mathcal{P}_x &= \{a = x_0 < x_1 < \cdots < x_n = b\}\,,\\ \mathcal{P}_y &= \{c = y_0 < y_1 < \cdots < y_m = d\}\,,\\ \mathcal{P}_z &= \{r = z_0 < z_1 < \cdots < z_p = s\}\,,\end{aligned}$$

be partitions of $[a, b]$, $[c, d]$, $[r, s]$, respectively, and $\mathcal{P}$ be a collection of non-overlapping cubes given by

$$\mathcal{P} = \left\{R_{ijk} \,\middle|\, R_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k], 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant p\right\}.$$

Such a collection $\mathcal{P}$ is called a partition of Q, and the norm of $\mathcal{P}$ is the maximum of the length of the diagonals of all R_{ijk}; that is

$$\|\mathcal{P}\| = \max\left\{\sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2 + (z_k - z_{k-1})^2} \,\middle|\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant p\right\}.$$

A Riemann sum of f for this partition $\mathcal{P}$ is given by

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} f(\xi_{ijk}, \eta_{ijk}, \zeta_{ijk})(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})\,,$$

where $\left\{(\xi_{ijk}, \eta_{ijk}, \zeta_{ijk})\right\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m, 1\leqslant k\leqslant p}$ is a collection of points satisfying $(\xi_{ijk}, \eta_{ijk}, \zeta_{ijk}) \in Q_{ijk}$ for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ and $1 \leqslant k \leqslant p$. The mass of Q then should be the "limit" of Riemann sums as $\|\mathcal{P}\|$ approaches zero. In general, we can remove the restrictions that f is non-negative on R and still consider the limit of the Riemann sums. We have the following

Theorem 14.14

Let $Q = [a, b] \times [c, d] \times [r, s]$ be a cube in the space, and $f : Q \to \mathbb{R}$ be a function. f is said to be Riemann integrable on Q if there exists a real number I such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of Q satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ belongs to $(I - \varepsilon, I + \epsilon)$. Such a number I (is unique if it exists and) is called the ***Riemann integral*** or ***triple integral of f on*** Q and is denoted by $\displaystyle\iiint\limits_Q f(x, y, z)\, dV$.