微積分 MA1002-A 上課筆記（精簡版）

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Suppose that $R = [a, b] \times [c, b] = \{(x, y) \,|\, a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$ is a rectangular region in the plane, $f : R \to \mathbb{R}$ is a non-negative continuous function, and D is the solid

$$\mathrm{D} = \{(x, y, z) \,|\, (x, y) \in R\,, 0 \leqslant z \leqslant f(x, y)\}\,.$$

Let $\mathcal{P}_x = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ and $\mathcal{P}_y = \{c = y_0 < y_1 < \cdots < y_m = d\}$ be partitions of $[a, b]$ and $[c, d]$, respectively. Let R_{ij} denote the rectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$. By relabeling the rectangles as $R_1, R_2, \cdots, R_{nm}$, the collection of rectangles $\mathcal{P} = \{R_k \,|\, 1 \leqslant k \leqslant nm\}$ is called a partition of R. Let $\{(\xi_k, \eta_k)\}_{k=1}^{nm}$ be a collection of point in R such that $(\xi_k, \eta_k) \in R_k$, the volume of the solid can be approximated by

$$\sum_{k=1}^{nm} f(\xi_k, \eta_k) A_k\,,$$

where A_k is the area of the rectangle R_k. The sum above is called a ***Riemann sum of*** f ***for partition*** $\mathcal{P}$. Define the norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, as the maximum length of the diagonal of R_k, then the volume of D is the "limit"

$$\lim_{\|\mathcal{P}\| \to 0} \sum_{k=1}^{nm} f(\xi_k, \eta_k) A_k$$

as long as "the limit exists". Similar to the discussion of the limit of Riemann sums in the case of functions of one variable, we can remove the restrictions that f is continuous and non-negative on R and still consider the limit of the Riemann sums. We have the following

Definition 14.1

Let $R = [a, b] \times [c, d]$ be a rectangle in the plane, and $f : R \to \mathbb{R}$ be a function. f is said to be Riemann integrable on R if there exists a real number V such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is partition of R satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums of f for the partition $\mathcal{P}$ belongs to the interval $(V - \varepsilon, V + \varepsilon)$. Such a number V (is unique if it exists and) is called the ***Riemann integral*** or ***double integral of*** f ***on*** R and is denoted by $\displaystyle\iint_R f(x, y)\, dA$.

How about the case that the base R of the solid is not a closed and bounded rectangle? In this case we choose $r > 0$ large enough such that $R \subseteq [-r, r]^2 \equiv [-r, r] \times [-r, r]$ and then for a function $f : R \to \mathbb{R}$, define $\widetilde{f} : [-r, r]^2 \to \mathbb{R}$ by

$$\widetilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in R\,, \\ 0 & \text{if } (x, y) \notin R\,. \end{cases}$$

We define $\iint\limits_R f(x,y)\,dA$ as $\iint\limits_{[-r,r]^2} \widetilde{f}(x,y)\,dA$ (when the latter double integral exists).

Before proceeding, let us talk about a special class of regions.

Definition 14.2

A region R is said to be have area if the constant function 1 is Riemann integrable on R. If R has area, then the area of R is defined as the integral $\iint\limits_R 1\,dA$.

The following theorem is an analogy of Theorem 4.10.

Theorem 14.3

Let R be a closed and bounded region in the plane, and $f : R \to \mathbb{R}$ be a function. If R has area and f is continuous on R, then f is Riemann integrable on R.

Similar to the properties for integrals of functions of one variable, we have the following

Theorem 14.4: Properties of double integrals

Let R be a closed and bounded region in the plane, $f, g : R \to \mathbb{R}$ be functions that are Riemann integrable on R, and c be a real number.

1. cf is Riemann integrable on R, and
$$\iint\limits_R (cf)(x,y)\,dA = c\iint\limits_R f(x,y)\,dA\,.$$
2. $f \pm g$ are Riemann integrable on R, and
$$\iint\limits_R (f \pm g)(x,y)\,dA = \iint\limits_R f(x,y)\,dA \pm \iint\limits_R g(x,y)\,dA\,.$$
3. If $f(x,y) \geqslant g(x,y)$ for all $(x,y) \in R$, then
$$\iint\limits_R f(x,y)\,dA \geqslant \iint\limits_R g(x,y)\,dA\,.$$
4. $|f|$ is Riemann integrable, and
$$\Big|\iint\limits_R f(x,y)\,dA\Big| \leqslant \iint\limits_R \big|f(x,y)\big|\,dA\,.$$

Definition 14.5

Two bounded regions R_1 and R_2 in the plane are said to be non-overlapping if $R_1 \cap R_2$ has zero area.

Theorem 14.6

Let R_1 and R_2 be non-overlapping regions in the plane, $R = R_1 \cup R_2$, and $f : R \to \mathbb{R}$ be such that f is Riemann integrable on R_1 and R_2. Then f is Riemann integrable on R and

$$\iint\limits_{R} f(x, y)\, dA = \iint\limits_{R_1} f(x, y)\, dA + \iint\limits_{R_2} f(x, y)\, dA\,.$$

14.2 The Iterated Integrals and Fubini's Theorem

Let R be a bounded region with area, and $f : R \to \mathbb{R}$ be a non-negative continuous function. As explained in the previous section, the volume of the solid

$$\mathrm{D} = \big\{(x, y, z)\,\big|\,(x, y) \in R, 0 \leqslant z \leqslant f(x, y)\big\}$$

is given by $\displaystyle\iint\limits_{R} f(x, y)\, dA$. We are concerned with computing this double integral in this section.

Recall from Section 7.2 that if D is a solid lies between two planes $x = a$ and $x = b$ $(a < b)$, and the area of the cross section of D taken perpendicular to the x-axis is $A(x)$, then

$$\text{the volume of D} = \int_a^b A(x)\, dx\,.$$

Therefore, if the region R is given by

$$R = \big\{(x, y)\,\big|\,a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\big\}$$

for some continuous functions $g_1, g_2 : [a, b] \to \mathbb{R}$, then the area of the cross section of D taken perpendicular to the x axis is

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

which shows that the volume of D is given by $\displaystyle\int_a^b \Big(\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\Big)\, dx$. Therefore, in this special case we find that

$$\iint\limits_{R} f(x, y)\, dA = \int_a^b \Big(\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\Big)\, dx\,. \tag{14.2.1}$$

Similarly, recall that if D lies between $y = c$ and $y = d$ ($c < d$), and the area of the cross section of D taken perpendicular to the y-axis is $A(y)$, then

$$\text{the volume of D} = \int_c^d A(y)\,dy\,;$$

thus similar argument shows that

$$\iint\limits_R f(x,y)\,dA = \int_c^d \Big(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \Big)\,dy\,. \tag{14.2.2}$$

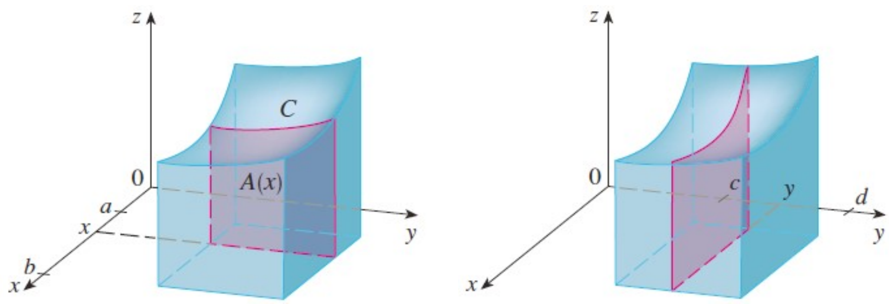

Figure 14.2: Finding the volume of D using the method of cross section

We note that in formulas (14.2.1), we have to compute the integral $\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$ for each fixed $x \in [a,b]$ which gives the area of the cross section $A(x)$, then compute the integral $\int_a^b A(x)\,dx$ to obtain the volume of D. This way of computing double integrals is called ***iterated integrals***, and sometime we omit the parentheses and write it as

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx\,.$$

Similarly, the iterated integral appearing in (14.2.2) can also be written as

$$\iint\limits_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy\,.$$

The evaluation of the double integral $\iint\limits_R f(x,y)\,dA$ can be generalized for a more general class of functions, and it is called the Fubini Theorem.

Theorem 14.7: Fubini's Theorem

Let R be a region in the plane, and $f : R \to \mathbb{R}$ be continuous (but no necessary non-negative).

1. If R is given by $R = \big\{(x, y) \,\big|\, a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\big\}$, then

$$\iint\limits_R f(x, y)\, dA = \int_a^b \Big(\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \Big)\, dx\,.$$

2. If R is given by $R = \big\{(x, y) \,\big|\, c \leqslant y \leqslant d, g_1(x) \leqslant y \leqslant g_2(x)\big\}$, then

$$\iint\limits_R f(x, y)\, dA = \int_c^d \Big(\int_{h_1(y)}^{h_2(y)} f(x, y)\, dx \Big)\, dy\,.$$

Example 14.8. Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane. By the definition of double integrals, the volume of this solid is given by $\iint\limits_R (4 - x^2 - 2y^2)\, dA$, where R is the region $\big\{(x, y) \,\big|\, x^2 + 2y^2 \leqslant 4\big\}$. Writing R as

$$R = \Big\{(x, y) \,\Big|\, -2 \leqslant x \leqslant 2\,, -\sqrt{\frac{4 - x^2}{2}} \leqslant y \leqslant \sqrt{\frac{4 - x^2}{2}}\Big\}$$

or

$$R = \big\{(x, y) \,\big|\, -\sqrt{2} \leqslant y \leqslant \sqrt{2}\,, -\sqrt{4 - 2y^2} \leqslant x \leqslant \sqrt{4 - 2y^2}\big\}\,,$$

the Fubini Theorem then implies that

$$\begin{aligned} \iint\limits_R (4 - x^2 - 2y^2)\, dA &= \int_{-2}^{2} \Big(\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (4 - x^2 - 2y^2)\, dy \Big)\, dx \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \Big(\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} (4 - x^2 - 2y^2)\, dx \Big)\, dy\,. \end{aligned}$$