微積分 MA1002-A 上課筆記（精簡版）

2019.05.21.

Ching-hsiao Arthur Cheng 鄭經斅

Theorem 13.74

Let f and g be continuously differentiable functions of n variables. Suppose that on the level curve $g(x_1, \cdots, x_n) = c$ the function f attains its extrema at $(a_1, \cdots, a_n)$. If $(\nabla g)(a_1, \cdots, a_n) \neq \mathbf{0}$, then there is a real value λ such that

$$(\nabla f)(a_1, \cdots, a_n) = \lambda(\nabla g)(a_1, \cdots, a_n)\,.$$

Theorem 13.76: Lagrange Multiplier Theorem - More General Version

Let f, g and h be continuously differentiable functions of three variables. Suppose that subject to the constraints $g(x, y, z) = c_1$ and $h(x, y, z) = c_2$ the function f attains its extrema at (x_0, y_0, z_0). If $(\nabla g)(x_0, y_0, z_0) \times (\nabla h)(x_0, y_0, z_0) \neq \mathbf{0}$, then there are real numbers λ and μ such that

$$(\nabla f)(x_0, y_0, z_0) = \lambda(\nabla g)(x_0, y_0, z_0) + \mu(\nabla h)(x_0, y_0, z_0)\,.$$

Example 13.77. Find the extreme value of the function $f(x, y, z) = 20 + 2x + 2y + z^2$ subject to two constraints $x^2 + y^2 + z^2 = 11$ and $x + y + z = 3$.

Let $g(x, y, z) = x^2 + y^2 + z^2 - 11$ and $h(x, y, z) = x + y + z - 3$. We first note that if (x, y, z) satisfies $g(x, y, z) = h(x, y, z) = 0$, then $(\nabla g)(x, y, z) \times (\nabla h)(x, y, z) \neq \mathbf{0}$. Moreover, f attains its extrema on the intersection of the level surface $g(x, y, z) = 0$ and $h(x, y, z) = 0$ (since the intersection is closed and bounded). Suppose that f attains its extrema at (x_0, y_0, z_0). Then there exists $\lambda, \mu \in \mathbb{R}$ such that

$$\begin{aligned}(\nabla f)(x_0, y_0, z_0) &= \lambda(\nabla g)(x_0, y_0, z_0) + \mu(\nabla h)(x_0, y_0, z_0)\,,\\ g(x_0, y_0, z_0) &= h(x_0, y_0, z_0) = 0\,.\end{aligned}$$

Therefore,

$$2\lambda x_0 + \mu = 2\,, \tag{13.10.2a}$$

$$2\lambda y_0 + \mu = 2\,, \tag{13.10.2b}$$

$$2(\lambda - 1) z_0 + \mu = 0\,, \tag{13.10.2c}$$

$$x_0^2 + y_0^2 + z_0^2 = 11\,, \tag{13.10.2d}$$

$$x_0 + y_0 + z_0 = 3\,. \tag{13.10.2e}$$

(13.10.2a,b) implies that $\lambda(x_0 - y_0) = 0$; thus $\lambda = 0$ or $x_0 = y_0$.

1. If $\lambda = 0$, then (13.10.2a) implies $\mu = 2$ and (13.10.2c) implies $\mu = 2z_0$. Therefore, $z_0 = 1$ which further shows $x_0^2 + y_0^2 = 10$ and $x_0 + y_0 = 2$. Then $(x_0, y_0) = (3, -1)$ or $(-1, 3)$. Therefore, when $\lambda = 0$,
$$(x_0, y_0, z_0) = (3, -1, 1) \quad \text{or} \quad (x_0, y_0, z_0) = (-1, 3, 1)\,.$$

2. If $x_0 = y_0$, then (13.10.2d,e) implies that $2x_0^2 + z_0^2 = 11$ and $2x_0 + z_0 = 3$. Therefore, $x_0 = y_0 = \dfrac{3 \pm 2\sqrt{3}}{3}$, $z_0 = \dfrac{3 \mp 4\sqrt{3}}{3}$.

Since $f(3, -1, 1) = f(-1, 3, 1) = 25$ and
$$f\Big(\frac{3+2\sqrt{3}}{3}, \frac{3+2\sqrt{3}}{3}, \frac{3-4\sqrt{3}}{3}\Big) = f\Big(\frac{3-2\sqrt{3}}{3}, \frac{3-2\sqrt{3}}{3}, \frac{3+4\sqrt{3}}{3}\Big) = \frac{91}{3}\,,$$
we conclude that the maximum and minimum value of f subject to $g = h = 0$ are $\dfrac{91}{3}$ and 25, respectively.

Example 13.78. Find the extreme value of $f(x, y, z) = z$ subject to the constraints $x^4 + y^4 - z^3 = 0$ and $y = z$.

Let $g(x, y, z) = x^4 + y^4 - z^3$ and $h(x, y, z) = y - z$. Then
$$(\nabla g)(x, y, z) = (4x^3, 4y^3, -3z^2) \qquad \text{and} \qquad (\nabla h)(x, y, z) = (0, 1, -1)$$
which implies that
$$(\nabla g)(x, y, z) \times (\nabla h)(x, y, z) = (3z^2 - 4y^3, 4x^3, 4x^3)\,.$$
Suppose the extreme value of f, under the constraints $g = h = 0$, occurs at (x_0, y_0, z_0).

1. If $(\nabla g)(x_0, y_0, z_0) \times (\nabla h)(x_0, y_0, z_0) = \mathbf{0}$, then $(x_0, y_0, z_0) = (0, 0, 0)$ and $f(0, 0, 0) = 0$.

2. If $(\nabla g)(x_0, y_0, z_0) \times (\nabla h)(x_0, y_0, z_0) \neq \mathbf{0}$, then the Lagrange Multiplier Theorem implies that there exist $\lambda, \mu \in \mathbb{R}$ such that
$$(\nabla f)(x_0, y_0, z_0) = \lambda(\nabla g)(x_0, y_0, z_0) + \mu(\nabla h)(x_0, y_0, z_0)\,.$$
Therefore, (x_0, y_0, z_0) satisfies that
$$4\lambda x_0^3 = 0\,, \tag{13.10.3a}$$
$$4\lambda y_0^3 + \mu = 0\,, \tag{13.10.3b}$$
$$-3\lambda z_0^2 - \mu = 1\,, \tag{13.10.3c}$$
$$x_0^4 + y_0^4 - z_0^3 = 0\,, \tag{13.10.3d}$$
$$y_0 - z_0 = 0\,. \tag{13.10.3e}$$

Then (13.10.3a) implies that $\lambda = 0$ or $x_0 = 0$.

(a) If $\lambda = 0$, then (13.10.3b) shows $\mu = 0$; thus using (13.10.3c), we obtain a contradiction $0 = -1$. Therefore, $\lambda \neq 0$.

(b) If $x_0 = 0$ (and $\lambda \neq 0$), then (13.10.3d) implies that $y_0^4 - z_0^3 = 0$. Together with (13.10.3e), we find that $y_0 = 0$ or $y_0 = 1$. However, if $y_0 = 0$, then (13.10.3b) shows that $\mu = 0$ which again implies a contradiction $0 = 1$ from (13.10.3c). Therefore, $y_0 = z_0 = 1$ (and there are λ, μ satisfying (13.10.3b,c) for $y_0 = z_0 = 1$ but the values of λ and μ are not important).

Therefore, the Lagrange Multiplier Theorem only provides one possible $(x_0, y_0, z_0) = (0, 1, 1)$ where f attains its extreme value.

Since the intersection of the level surface $g = 0$ and $h = 0$ is closed and bounded, f must attains its maximum and minimum subject to the constraints $g = h = 0$. Since $(0, 0, 0)$ and $(0, 1, 1)$ are the only possible points where f attains its extrema, the maximum and minimum of f, subject to the constraint $g = h = 0$, is $f(0, 1, 1) = 1$ and $f(0, 0, 0) = 0$, respectively.