微積分 MA1002-A 上課筆記（精簡版）

2019.05.16.

Ching-hsiao Arthur Cheng 鄭經斅

Theorem 13.41: Implicit Function Theorem (Special case)

Let F be a function of n variables $(x_1, x_2, \cdots, x_n)$ such that F_{x_1}, F_{x_2}, $\cdots$, F_{x_n} are continuous in a neighborhood of $(a_1, a_2, \cdots, a_n$. If $F(a_1, a_2, \cdots, a_n) = 0$ and $F_{x_n}(a_1, a_2, \cdots, a_n) \neq 0$, then locally near $(a_1, a_2, \cdots, a_n)$ there exists a unique continuous function f satisfying $F(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1})) = 0$ and $a_n = f(a_1, \cdots, a_{n-1})$. Moreover, for $1 \leqslant j \leqslant n-1$,

$$\frac{\partial f}{\partial x_j}(x_1, \cdots, x_{n-1}) = -\frac{F_{x_j}(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1}))}{F_{x_n}(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1}))}\,.$$

Theorem 13.69: Lagrange Multiplier Theorem - Simplest Version

Let f and g be continuously differentiable functions of two variables. Suppose that on the level curve $g(x, y) = c$ the function f attains its extrema at (x_0, y_0). If $(\nabla g)(x_0, y_0) \neq \mathbf{0}$, then there is a real value λ such that

$$(\nabla f)(x_0, y_0) = \lambda(\nabla g)(x_0, y_0)\,.$$

Remark 13.70. The scalar λ in the theorem above is called a Lagrange multiplier.

Proof of Theorem 13.69. First we note that (x_0, y_0) is on the level curve $g(x, y) = c$; thus $c = g(x_0, y_0)$.

Define $F(x, y) = g(x, y) - g(x_0, y_0)$. Then F has continuous first partial derivatives, and $(\nabla F)(x_0, y_0) = (\nabla g)(x_0, y_0) \neq \mathbf{0}$. Then either $F_x(x_0, y_0) \neq 0$ or $F_y(x_0, y_0) \neq 0$. Suppose that $F_y(x_0, y_0) \neq 0$. Then the Implicit Function Theorem implies that there exist $\delta > 0$ and a unique differentiable function $h : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$ such that

$$F(x, h(x)) = 0 \qquad \text{and} \qquad y_0 = h(x_0)\,.$$

In other words, the set $\{(x, h(x)) \,|\, x_0 - \delta < x < x_0 + \delta\}$ is a subset of the level curve $g(x, y) = g(x_0, y_0)$. Therefore, the function $G : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$ defined by $G(x) = f(x, h(x))$ attains its extrema at (an interior point) x_0; thus

$$G'(x_0) = f_x(x_0, y_0) + f_y(x_0, y_0)h'(x_0) = 0\,.$$

Since the implicit differentiation shows that

$$h'(x_0) = -\frac{F_x(x_0, h(x_0))}{F_y(x_0, h(x_0))} = -\frac{g_x(x_0, y_0)}{g_y(x_0, y_0)}\,,$$

we conclude that

$$f_x(x_0, y_0) - f_y(x_0, y_0)\frac{g_x(x_0, y_0)}{g_y(x_0, y_0)} = 0\,.$$

If $f_y(x_0, y_0) = 0$, then $f_x(x_0, y_0) = 0$ which implies that $(\nabla f)(x_0, y_0) = \mathbf{0} = 0 \cdot (\nabla g)(x_0, y_0)$. If $f_y(x_0, y_0) \neq 0$, then

$$\frac{f_x(x_0, y_0)}{f_y(x_0, y_0)} = \frac{g_x(x_0, y_0)}{g_y(x_0, y_0)}$$

which implies that $(\nabla f)(x_0, y_0) /\!/ (\nabla g)(x_0, y_0)$; thus there exists λ such that

$$(\nabla f)(x_0, y_0) = \lambda(\nabla g)(x_0, y_0)\,.$$

Similar argument can be applied to the case $F_x(x_0, y_0) \neq 0$, and we omit the proof for this case. □

Example 13.71. Find the extreme value of $f(x, y) = 4xy$ subject to the constraint

$$\frac{x^2}{9} + \frac{y^2}{16} = 1\,.$$

Let $g(x, y) = \frac{x^2}{9} + \frac{y^2}{16} - 1$. Suppose that on the level curve $g(x, y) = 0$ the function f attains its extrema at (x_0, y_0). Note that then $(\nabla g)(x_0, y_0) \neq \mathbf{0}$ (since $(x_0, y_0) \neq (0, 0)$); thus the Lagrange Multiplier Theorem implies that there exists $\lambda \in \mathbb{R}$ such that

$$(4y_0, 4x_0) = (\nabla f)(x_0, y_0) = \lambda(\nabla g)(x_0, y_0) = \lambda\Big(\frac{2x_0}{9}, \frac{y_0}{8}\Big)\,.$$

Therefore, (x_0, y_0) satisfies $4y_0 = \frac{2\lambda x_0}{9}$ and $4x_0 = \frac{\lambda y_0}{8}$, as well as $\frac{x_0^2}{9} + \frac{y_0^2}{16} = 1$. Therefore, $\lambda \neq 0$, and

$$4x_0 = \frac{\lambda y_0}{8} = \frac{\lambda}{8} \cdot \frac{\lambda x_0}{18} = \frac{\lambda^2 x_0}{144}\,.$$

The identity above implies that $x_0 = 0$ or $\lambda = \pm 24$.

1. If $x_0 = 0$, then $y_0 = \pm 4$ which shows that $\lambda = 0$, a contradiction.

2. If $\lambda = \pm 24$, then $x_0 = \pm\frac{3y_0}{4}$; thus

$$1 = \frac{1}{9} \cdot \frac{9y_0^2}{16} + \frac{y_0^2}{16} = \frac{y_0^2}{8}\,.$$

   Therefore, $y_0 = \pm 2\sqrt{2}$ which implies that $x_0 = \pm\frac{3\sqrt{2}}{2}$. At these (x_0, y_0), $f(x_0, y_0) = \pm 24$. Therefore, on the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ the maximum of f is 24 $\Big($at $(x_0, y_0) = \big(\pm 2\sqrt{2}, \pm\frac{3\sqrt{2}}{2}\big)\Big)$ and the minimum of f is -24 $\Big($at $(x_0, y_0) = \big(\pm 2\sqrt{2}, \mp\frac{3\sqrt{2}}{2}\big)\Big)$.

Example 13.72. Find the extreme value of $f(x,y) = 4xy$, where $x > 0$ and $y > 0$, subject to the constraint $\frac{x^2}{9} + \frac{y^2}{16} = 1$. From the previous example we find that the maximum of f is 24 $\Big($at $(x_0, y_0) = \big(2\sqrt{2}, \frac{3\sqrt{2}}{2}\big)\Big)$. The minimum of f occurs at the end-points $(0,4)$ or $(3,0)$. In either points, the value of f is 0; thus the minimum of f is 0.

Example 13.73. Find the extreme value of $f(x,y) = 4xy$, where (x,y) satisfies $\frac{x^2}{9} + \frac{y^2}{16} \leqslant 1$. We have find the extreme value of f, under the constraint $\frac{x^2}{9} + \frac{y^2}{16} = 1$, is ± 24. Therefore, it suffices to consider the extreme value of f in the interior $\frac{x^2}{9} + \frac{y^2}{16} < 1$.

Assume that f attains its extreme value at an interior point (x_0, y_0). Then (x_0, y_0) is a critical point of f; thus

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$$

which implies that $(x_0, y_0) = (0, 0)$. Since $f(0,0) = 0$, $f(0,0)$ is not an extreme value of f. Therefore, the extreme value of f on the region $\frac{x^2}{9} + \frac{y^2}{16} \leqslant 1$ is ± 24.

We note that $(0,0)$ in fact is a saddle point of f since $f_{xx}(0,0) f_{yy}(0,0) - f_{xy}(0,0)^2 = -16 < 0$.

Similar argument of proving Theorem 13.69 can be used to show the following

Theorem 13.74

Let f and g be continuously differentiable functions of n variables. Suppose that on the level curve $g(x_1, \cdots, x_n) = c$ the function f attains its extrema at $(a_1, \cdots, a_n)$. If $(\nabla g)(a_1, \cdots, a_n) \neq \mathbf{0}$, then there is a real value λ such that

$$(\nabla f)(a_1, \cdots, a_n) = \lambda (\nabla g)(a_1, \cdots, a_n) .$$

Example 13.75. Find the minimum value of $f(x,y,z) = 2x^2 + y^2 + 3z^2$ subject to the constraint $2x - 3y - 4z = 49$.

Let $g(x,y,z) = 2x - 3y - 4z - 49$. Then $(\nabla g) \neq \mathbf{0}$; thus if f attains its relative extrema at (x_0, y_0, z_0), there exists $\lambda \in \mathbb{R}$ such that $(\nabla f)(x_0, y_0, z_0) = \lambda (\nabla g)(x_0, y_0, z_0)$. Therefore,

$$(4x_0, 2y_0, 6z_0) = \lambda(2, -3, -4)$$

or equivalently, $\lambda = 2x_0 = -\frac{2}{3}y_0 = -\frac{3}{2}z_0$. Since $2x_0 - 3y_0 - 4z_0 = 49$, we find that $\lambda = 6$ which implies that

$$(x_0, y_0, z_0) = (3, -9, -4) .$$

Since f grows beyond any bound as $\sqrt{x^2+y^2+z^2}$ approaches ∞, we find that $f(3,-9,-4) = 147$ is the minimum of f.

Next, we consider the optimization problem of finding the extreme value of a function of three variables $w = f(x,y,z)$ subject to two constraints $g(x,y,z) = h(x,y,z) = 0$.

Theorem 13.76: Lagrange Multiplier Theorem - More General Version

Let f, g and h be continuously differentiable functions of three variables. Suppose that subject to the constraints $g(x,y,z) = c_1$ and $h(x,y,z) = c_2$ the function f attains its extrema at (x_0,y_0,z_0). If $(\nabla g)(x_0,y_0,z_0) \times (\nabla h)(x_0,y_0,z_0) \neq \mathbf{0}$, then there are real numbers λ and μ such that

$$(\nabla f)(x_0,y_0,z_0) = \lambda(\nabla g)(x_0,y_0,z_0) + \mu(\nabla h)(x_0,y_0,z_0)\,.$$