微積分 MA1002-A 上課筆記（精簡版）

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Theorem 13.41: Implicit Function Theorem (Special case)

Let F be a function of n variables $(x_1, x_2, \cdots, x_n)$ such that F_{x_1}, F_{x_2}, $\cdots$, F_{x_n} are continuous in a neighborhood of $(a_1, a_2, \cdots, a_n$. If $F(a_1, a_2, \cdots, a_n) = 0$ and $F_{x_n}(a_1, a_2, \cdots, a_n) \neq 0$, then locally near $(a_1, a_2, \cdots, a_n)$ there exists a unique continuous function f satisfying $F(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1})) = 0$ and $a_n = f(a_1, \cdots, a_{n-1})$. Moreover, for $1 \leqslant j \leqslant n-1$,

$$\frac{\partial f}{\partial x_j}(x_1, \cdots, x_{n-1}) = -\frac{F_{x_j}(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1}))}{F_{x_n}(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1}))}\,.$$

Definition 13.50

Let f be a function of n variables. The directional derivative of f at $(a_1, a_2, \cdots, a_n)$ in the direction $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)$, where $u_1^2 + u_2^2 + \cdots + u_n^2 = 1$, is the limit

$$(D_{\boldsymbol{u}} f)(a_1, a_2, \cdots, a_n) = \lim_{h \to 0} \frac{f(a_1 + hu_1, a_2 + hu_2, \cdots, a_n + hu_n) - f(a_1, a_2, \cdots, a_n)}{h}$$

provided that the limit exists. The gradient of f at $(a_1, a_2, \cdots, a_n)$, denoted by $(\nabla f)(a_1, a_2, \cdots, a_n)$, is the vector

$$(\nabla f)(a_1, a_2, \cdots, a_n) = \big(f_{x_1}(a_1, \cdots, a_n), f_{x_2}(a_1, \cdots, a_n), \cdots, f_{x_n}(a_1, \cdots, a_n)\big)\,.$$

Theorem 13.51

Let f be a function of n variables. If f is differentiable at $(a_1, a_2, \cdots, a_n)$ and $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)$ is a unit vector, then

$$(D_{\boldsymbol{u}} f)(a_1, a_2, \cdots, a_n) = (\nabla f)(a_1, \cdots, a_n) \cdot \boldsymbol{u}\,.$$

13.7 Tangent Planes and Normal Lines

- **The tangent plane of surfaces**

Any three points in the space that are not collinear defines a plane. Suppose that $\mathcal{S}$ is a "surface" (which we have not define yet, but please use the common sense to think about it), and $P_0 = (x_0, y_0, z_0)$ is a point on the plane. Given another two point $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ on the surface such that P_0, P_1, P_2 are not collinear, let $T_{P_1P_2}$ denote the plane determined by P_0, P_1 and P_2. If the plane "approaches" a certain plane as P_1, P_2 approaches P_0, the "limit" is called the tangent plane of $\mathcal{S}$ at P_0.

Now suppose that the surface $\mathcal{S}$ is the graph of a function of two variables $z = f(x, y)$. Consider the tangent plane of $\mathcal{S}$ at $P_0 = (x_0, y_0, z_0)$, where $z_0 = f(x_0, y_0)$. For $h, k \neq 0$, let $P_1 = (x_0 + h, y_0, f(x_0 + h, y_0))$ and $P_2 = (x_0, y_0 + k, f(x_0, y_0 + k))$, as well as

$$\boldsymbol{u} = \Big(1, 0, \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}\Big) \qquad \text{and} \qquad \boldsymbol{v} = \Big(0, 1, \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}\Big) .$$

Then the plane $T_{P_1P_2}$ is given by

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot (x - x_0, y - y_0, z - z_0) = 0 ,$$

where $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \times \boldsymbol{v}$ are the inner product and the cross product of $\boldsymbol{u}$ and $\boldsymbol{v}$ defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = (u_1v_1 + u_2v_2 + u_3v_3) \quad \text{and} \quad \boldsymbol{u} \times \boldsymbol{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) ,$$

respectively. In other words, the plane $T_{P_1P_2}$ is given by

$$\Big(-\frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}, -\frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}, 1\Big) \cdot (x - x_0, y - y_0, z - z_0) = 0 .$$

Suppose that f is differentiable at (x_0, y_0). Passing to the limit as $(h, k) \to (0, 0)$, we find that the limit is

$$\big(-f_x(x_0, y_0), -f_y(x_0, y_0), 1\big) \cdot \big(x - x_0, y - y_0, z - f(x_0, y_0)\big) = 0$$

or equivalently (using $z_0 = f(x_0, y_0)$),

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) .$$

On the other hand, if f is differentiable at (x_0, y_0), then

$$\begin{aligned} f(x, y) = f(x_0, y_0) &+ f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &+ \varepsilon_1(x, y)(x - x_0) + \varepsilon_2(x, y)(y - y_0) \end{aligned}$$

for some functions ε_1, ε_2 satisfying $\lim\limits_{(x,y)\to(x_0,y_0)} \varepsilon_1(x, y) = \lim\limits_{(x,y)\to(x_0,y_0)} \varepsilon_2(x, y) = 0$. This shows that the rate of convergence of the quantity

$$\big|f(x, y) - f(x_0, y_0) - f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0)\big| ,$$

as (x, y) approaches (x_0, y_0), is "faster than linear" and this is exactly what we have in mind when talking about tangent planes. Therefore, we conclude that

> Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. If f is differentiable at $(x_0, y_0) \in R$, the tangent plane of the graph of f at $(x_0, y_0, f(x_0, y_0))$ is given by
>
> $$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)\,,$$
>
> and the vector $(f_x(x_0, y_0), f_y(x_0, y_0), -1)$ is a normal vector to the graph of f at $(x_0, y_0, f(x_0, y_0))$.

Now suppose that the function of three variables $w = F(x, y, z)$ is continuously differentiable; that is, F_x, F_y, F_z are continuous. Suppose that for some (x_0, y_0, z_0) in the domain, $(\nabla F)(x_0, y_0, z_0) \neq \mathbf{0}$. W.L.O.G., we assume that $F_z(x_0, y_0, z_0) \neq 0$. Define

$$G(x, y, z) = F(x, y, z) - F(x_0, y_0, z_0)\,.$$

Then $G_x = F_x$, $G_y = F_y$ and $G_z = F_y$, and the Implicit Function Theorem (Theorem 13.41) implies that there exists a unique differentiable function $z = f(x, y)$ such that

$$G(x, y, f(x, y)) = 0 \qquad \text{and} \qquad z_0 = f(x_0, y_0)\,.$$

In other words, the graph of f is a subset of the level surface $F(x, y, z) = F(x_0, y_0, z_0)$. By the discussion above, the tangent plane of the graph of f at (x_0, y_0, z_0) is given by

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

and the implicit partial differentiation further shows that the tangent plane above can be rewritten as

$$z = z_0 - \frac{F_x(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)}(x - x_0) - \frac{F_y(x_0, y_0, z_0)}{F_z(x_0, y_0, z_0)}(y - y_0)\,.$$

Therefore, the tangent plane of the graph of f at (x_0, y_0, z_0) is given by

$$(\nabla F)(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0\,.$$

On the other hand, note that the graph of f is the same as the level surface $F(x, y, z) =$

$F(x_0, y_0, z_0)$; thus we conclude that

> Let $w = F(x,y,z)$ be a function of three variables such that F_x, F_y and F_z are continuous. If $(\nabla F)(x_0,y_0,z_0) \neq \mathbf{0}$, then the tangent plane of the level surface $F(x,y,z) = F(x_0,y_0,z_0)$ at (x_0,y_0,z_0) is given by
>
> $$(\nabla F)(x_0,y_0,z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0\,,$$
>
> and the vector $(\nabla F)(x_0,y_0,z_0)$ is a normal vector to the level surface $F(x,y,z)$ $= F(x_0,y_0,z_0)$.

- **Properties of the gradient**

Theorem 13.52

Let F be a function of three variables. If F has continuous first partial derivatives F_x, F_y, F_z in a neighborhood of (x_0,y_0,z_0) and $(\nabla F)(x_0,y_0,z_0) \neq \mathbf{0}$, then $(\nabla F)(x_0,y_0,z_0)$ is perpendicular/normal to the level surface $F(x,y,z) = F(x_0,y_0,z_0)$ at (x_0,y_0,z_0). Moreover, the value of F at (x_0,y_0,z_0) increase most rapidly in the direction $\dfrac{(\nabla F)(x_0,y_0,z_0)}{\|(\nabla F)(x_0,y_0,z_0)\|}$ and decreases most rapidly in the direction $-\dfrac{(\nabla F)(x_0,y_0,z_0)}{\|(\nabla F)(x_0,y_0,z_0)\|}$, where $\|\cdot\|$ denotes the length of the vector.

Remark 13.53. The terminology "the value of f at (x_0,y_0,z_0) increase most rapidly in the direction $\boldsymbol{u}$", where $\boldsymbol{u}$ is a unit vector, means that the directional derivative $(D_{\boldsymbol{v}}f)(x_0,y_0,z_0)$, treated as a function of $\boldsymbol{v}$, attains its maximum at $\boldsymbol{v} = \boldsymbol{u}$.

Proof of Theorem 13.52. We have shown that $(\nabla F)(x_0,y_0,z_0)$ is perpendicular to the level surface $F(x,y,z) = F(x_0,y_0,z_0)$, so it suffices to show that $(D_{\boldsymbol{v}}F)(x_0,y_0,z_0)$ attains its maximum at $\boldsymbol{v} = \boldsymbol{u}$. Nevertheless, by Theorem 13.51, we find that

$$(D_{\boldsymbol{v}}F)(x_0,y_0,z_0) = (\nabla F)(x_0,y_0,z_0) \cdot \boldsymbol{v} = \|(\nabla F)(x_0,y_0,z_0)\| \cos\theta\,,$$

where θ is the angle between $(\nabla F)(x_0,y_0,z_0)$ and $\boldsymbol{v}$. Clearly $(D_{\boldsymbol{v}}F)(x_0,y_0,z_0)$ attains its maximum when $\theta = 0$ which shows that $(D_{\boldsymbol{v}}F)(x_0,y_0,z_0)$ attains its maximum at $\boldsymbol{v} = \dfrac{(\nabla F)(x_0,y_0,z_0)}{\|(\nabla F)(x_0,y_0,z_0)\|}$. □

Similarly, for functions of two variables, we have the following

Theorem 13.54

Let f be a function of two variables. If f has continuous first partial derivatives f_x and f_y in a neighborhood of (x_0, y_0) and $(\nabla f)(x_0, y_0) \neq \mathbf{0}$, then $(\nabla f)(x_0, y_0)$ is perpendicular/normal to the level curve $f(x, y) = f(x_0, y_0)$ at (x_0, y_0). Moreover, the value of f at (x_0, y_0) increase most rapidly in the direction $\dfrac{(\nabla f)(x_0, y_0)}{\|(\nabla f)(x_0, y_0)\|}$ and decreases most rapidly in the direction $-\dfrac{(\nabla f)(x_0, y_0)}{\|(\nabla f)(x_0, y_0)\|}$, where $\|\cdot\|$ denotes the length of the vector.

Example 13.55. Find an equation of the normal line and the tangent plane to the paraboloid

$$z = 1 - \frac{1}{10}(x^2 + 4y^2)$$

at the point $\big(1, 1, \frac{1}{2}\big)$.

Let $F(x, y, z) = z - 1 + \frac{1}{10}(x^2 + 4y^2)$. Then $F_z\big(1, 1, \frac{1}{2}\big) \equiv \big(\frac{1}{5}, \frac{4}{5}, 1\big) \neq \mathbf{0}$; thus Theorem 13.52 implies that the tangent plane of the given paraboloid at $\big(1, 1, \frac{1}{2}\big)$ is

$$z = \frac{1}{2} - \frac{1}{5}(x - 1) - \frac{4}{5}(y - 1) = \frac{3}{2} - \frac{1}{5}x - \frac{4}{5}y\,.$$

An equation of the normal line at $\big(1, 1, \frac{1}{2}\big)$ is given by

$$\frac{x - 1}{1/5} = \frac{y - 1}{4/5} = \frac{z - 1/2}{1}\,.$$