微積分 MA1002-A 上課筆記（精簡版）

2019.04.11.

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Chapter 13

Functions of Several Variables

13.1 Introduction to Functions of Several Variables

Definition 13.1

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is a real-valued function of (two variables) x and y. The set D is the domain of f, and the corresponding set of values for $f(x, y)$ is the range of f. For the function $z = f(x, y)$, x and y are called the independent variables and z is called the dependent variable.

Definition 13.2

Let f, g be real-valued functions of two variables with domain D.

1. The sum of f and g, the difference of f and g and the product of f and g, denoted by $f + g$, $f - g$ and fg, are functions defined on D given by

$$\begin{aligned}(f+g)(x,y) &= f(x,y) + g(x,y) \quad \forall\,(x,y) \in D\,,\\ (f-g)(x,y) &= f(x,y) - g(x,y) \quad \forall\,(x,y) \in D\,,\\ (fg)(x,y) &= f(x,y)g(x,y) \qquad \forall\,(x,y) \in D\,.\end{aligned}$$

2. The quotient of f and g, denoted by $\dfrac{f}{g}$, is a function defined on $D \backslash \big\{(x,y) \in D \,\big|\, g(x,y) = 0\big\}$ given by

$$\frac{f}{g}(x,y) = \frac{f(x,y)}{g(x,y)} \qquad \forall\,(x,y) \in D \text{ such that } g(x,y) \neq 0\,.$$

Remark 13.3. A function f of two variables should be given along with its domain. When the domain of a function is not specified, as before the domain should be treated as the collection of all (x, y) such that $f(x, y)$ is meaningful.

Definition 13.4

Let h be a real-valued function of two variables with domain D, and $g : I \to \mathbb{R}$ be a real-valued function (of one variable) on an interval I. The composite function of g and h, denoted by $g \circ h$, is a function defined on $D \cap \big\{(x, y) \in D \,\big|\, h(x, y) \in I\big\}$ given by

$$(g \circ h)(x, y) = g\big(h(x, y)\big) \qquad \forall\, (x, y) \in D \text{ such that } h(x, y) \in I\,.$$

Similar concepts such as real-valued functions of three variables, the sum, different, product, quotient and composition of functions of three variables can be defined accordingly.

Definition 13.5

Let D be a set of ordered pairs of real numbers, and $f : D \to \mathbb{R}$ be a real-valued function of two variables. The graph of f is the set of all points (x, y, z) for which $z = f(x, y)$ and $(x, y) \in D$.

Example 13.5. Let $r > 0$ be a real number. The graph of the function $z = f(x, y) = \sqrt{r^2 - x^2 - y^2}$ is the upper hemi-sphere of the sphere centered at the origin with radius r. On the other hand, the graph of the function $z = g(x, y) = -\sqrt{r^2 - x^2 - y^2}$ is the lower hemi-sphere of the sphere.