微積分 MA1002-A 上課筆記（精簡版）

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Theorem 9.76: Taylor's Theorem

Let $f : (a,b) \to \mathbb{R}$ be $(n+1)$-times differentiable, and $c \in (a,b)$. Then for each $x \in (a,b)$, there exists ξ between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)\,, \quad (9.7.1)$$

where Lagrange form of the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}\,.$$

Using Taylor's Theorem we also show that

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots \quad \forall\, x \in (0,1]\,.$$

Definition 9.80: Power Series

Let c be a real number. A power series (of one variable x) centered at c is an infinite series of the form

$$\sum_{k=0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + \cdots\,,$$

where a_k is independent of x and is called the coefficient of the k-th term.

Theorem 9.81

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers. If $\sum\limits_{k=0}^{\infty} a_k d^k$ converges, then $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges absolutely for all $x \in (c-|d|, c+|d|)$.

Definition 9.82: Radius of Convergence and Interval of Convergence

Let a power series centered at c be given. The radius of convergence of a power series centered at c is the greatest lower bound of the set

$$\big\{r > 0 \,\big|\, \text{there exists } x \in (c-r, c+r) \text{ such that the power series diverges}\big\}\,.$$

The set of all values of x for which the power series converges is called the interval of convergence of the power series.

Theorem 9.92: Properties of Functions Defined by Power Series

Suppose that the function f defined by power series

$$f(x)=\sum_{k=0}^{\infty}a_k(x-c)^k=a_0+a_1(x-c)+a_2(x-c)^2+\cdots$$

has a radius of convergence of $R>0$. Then

1. f is differentiable on $(c-R,c+R)$ and
$$f'(x)=\sum_{k=1}^{\infty}ka_k(x-c)^{k-1}=a_1+2a_2(x-c)+3a_3(x-c)^2+\cdots.$$
2. an anti-derivative of f on $(c-R,c+R)$ is given by
$$\int f(x)\,dx=C+\sum_{k=0}^{\infty}\frac{a_k}{k+1}(x-c)^{k+1}=C+a_0(x-c)+\frac{a_1}{2}(x-c)^2+\cdots.$$

The radius of convergence of the power series obtained by differentiating or integrating a power series term by term is the same as the original power series.

Corollary 9.95

For a function defined by power series

$$f(x)=\sum_{k=0}^{\infty}a_k(x-c)^k$$

(on a certain interval of convergence), the n-th Taylor polynomial for f at c is the n-th partial sum $\sum\limits_{k=0}^{n}a_k(x-c)^k$ of the power series.

9.9 Representation of Functions by Power Series

- **Geometric Power Series**

Recall that the geometric series $\sum\limits_{k=0}^{\infty}r^k$ converges if and only if $|r|<1$. The function $g(x)=\dfrac{1}{1-x}$ is defined on $\mathbb{R}\backslash\{1\}$, and by the fact that

$$\frac{1-x^{n+1}}{1-x}=1+x+x^2+\cdots+x^n=\sum_{k=0}^{n}x^k\qquad\forall\,x\neq 1,$$

we find that if $|x| < 1$, then

$$\lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} = \frac{1}{1 - x};$$

thus $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ on $(-1, 1)$. Therefore, for $c \neq b$,

$$\frac{1}{b - x} = \frac{1}{b - c} \cdot \frac{1}{1 - \frac{x - c}{b - c}} = \frac{1}{b - c} \sum_{k=0}^{\infty} \Big(\frac{x - c}{b - c}\Big)^k \qquad \forall\, x \text{ satisfying } \Big|\frac{x - c}{b - c}\Big| < 1\,,$$

or equivalently,

$$\frac{1}{b - x} = \sum_{k=0}^{\infty} \frac{1}{(b - c)^{k+1}} (x - c)^k \qquad \forall\, x \in (c - |b - c|, c + |b - c|)\,.$$

Replacing x by $-x$, we find that

$$\frac{1}{b + x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(b - c)^{k+1}} (x + c)^k \qquad \forall\, x \in (-c - |b - c|, -c + |b - c|)\,.$$

Example 9.96. Find a power series representation for $f(x) = \frac{1}{x}$, centered at 1.

To find the power series centered at 1, we rewrite $\frac{1}{x} = \frac{1}{1 + (x - 1)}$; thus

$$\frac{1}{x} = \frac{1}{1 - (1 - x)} = \sum_{k=0}^{\infty} (1 - x)^k = \sum_{k=0}^{\infty} (-1)^k (x - 1)^k \qquad \forall\, |x - 1| < 1\,.$$

Example 9.97. Find a power series representation for $f(x) = \ln x$ centered at 1.

Note that $\frac{d}{dx} \ln x = \frac{1}{x}$; thus

$$\frac{d}{dx} \ln x = \sum_{k=0}^{\infty} (-1)^k (x - 1)^k \qquad \forall\, x \in (0, 2)\,.$$

Therefore, by Theorem 9.92,

$$\ln x = C + \sum_{k=0}^{\infty} \frac{(-1)^k}{k + 1} (x - 1)^{k+1} = C + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (x - 1)^k \qquad \forall\, x \in (0, 2)\,.$$

To determine the constant C, we let $x = 1$ and find that $\ln 1 = C$; thus $C = 0$ and we conclude that

$$\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (x-1)^k \qquad \forall\, x \in (0,2)\,.$$

We note that the power series converges at $x = 2$, and Example 9.78 shows that

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\,.$$

In other words, the power series $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}(x-1)^k$ is continuous at 2.

- **Operations with Power Series**

Let $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ have interval of convergence I_1 and $g(x) = \sum_{k=0}^{\infty} b_k (x-c)^k$ have interval of convergence I_2.

1. $f(\alpha x) = \sum_{k=0}^{\infty} a_k \alpha^k \big(x - \frac{c}{\alpha}\big)^k$ on $I \equiv \big\{x \in \mathbb{R} \,\big|\, \alpha x \in I_1\big\}$.

2. $f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$ on $I \equiv I_1 \cap I_2$.

3. If $c = 0$ and $N \in \mathbb{N}$, then $f(x^N) = \sum_{k=0}^{\infty} a_k x^{Nk}$ on $I \equiv \big\{x \in \mathbb{R} \,\big|\, x^N \in I_1\big\}$.

4. $f(x)g(x) = \sum_{k=0}^{\infty} d_k (x-c)^k$ on $I \equiv I_1 \cap I_2$, where $d_k = \sum_{j=0}^{k} a_k b_{j-k}$.

Example 9.98. Find a power series for $f(x) = \arctan x$ centered at 0.

Note that $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$; thus

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} \qquad \forall\, x \in (-1,1)\,.$$

By Theorem 9.92,

$$\arctan x = C + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \qquad \forall\, x \in (-1,1)\,,$$

and the constant C is determined by applying the identity above at $x = 0$; thus $C = \arctan 0$ and

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \qquad \forall\, x \in (-1, 1)\,,$$

We note that the power series converges at $x = \pm 1$. Is it true that $\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$?

In general, suppose that the function f defined by power series $\sum_{k=0}^{\infty} a_k (x-c)^k$ has a radius of convergence $R > 0$, and g is a continuous function defined on some interval I such that $f(x) = g(x)$ for all $x \in (c-R, c+R) \subsetneq I$. If f is also defined on $c+R$ (or $c-R$), by Theorem 9.92 it is not clear if $\lim_{x \to c+R} f(x) = g(c+R)$ $\big($or $\lim_{x \to c-R} f(x) = g(c-R)\big)$. The following theorem concerns with this issue.

Theorem 9.97: Continuity of Power Series at End-points

Let the radius of convergence of the power series $f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ be r for some $r > 0$.

1. If $\sum_{k=0}^{\infty} a_k r^k$ converges, then f is continuous at $c + r$.
2. If $\sum_{k=0}^{\infty} a_k (-r)^k$ converges, then f is continuous at $c - r$.

Therefore, it is true that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots + \frac{(-1)^n}{2n+1} + \cdots.$$

9.10 Taylor and Maclaurin Series

Definition 9.98

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k$$

is called the Taylor series for f at c. It is also called the Maclaurin series for f if $c = 0$.

Theorem 9.99

Let f be a function that has derivatives of all orders at $x = c$, and P_n be the n-th Taylor polynomial for f at c. If R_n, the remainder associated with P_n, has the property that

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \forall\, x \in I$$

for some interval I, then the Taylor series for f converges and equals $f(x)$; that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \qquad \forall\, x \in I\,.$$

Corollary 9.100

Let f be a function that has derivatives of all orders in an open interval I containing c. If there exists $M > 0$ such that $\big|f^{(k)}(x)\big| \leqslant M$ for all $x \in I$ and each $k \in \mathbb{N}$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \qquad \forall\, x \in I\,.$$

Proof. By the Taylor Theorem,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x)\,,$$

where

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

for some ξ between c and x. Since $\big|f^{(k)}(x)\big| \leqslant M$ for all $x \in I$ and $k \in \mathbb{N}$, we find that

$$\big|R_n(x)\big| \leqslant \frac{M}{(n+1)!}|x-c|^{n+1} \qquad \forall\, x \in I\,.$$

Therefore, by the fact that $\displaystyle\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$ (the same reasoning as in Example 9.75), the Squeeze Theorem implies that

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \forall\, x \in I$$

and Theorem 9.99 further shows that $f(x) = \sum\limits_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$. □