微積分 MA1002-A 上課筆記（精簡版）

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Definition 9.80: Power Series

Let c be a real number. A power series (of one variable x) centered at c is an infinite series of the form

$$\sum_{k=0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + \cdots,$$

where a_k is independent of x and is called the coefficient of the k-th term.

Theorem 9.81

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers. If $\sum\limits_{k=0}^{\infty} a_k d^k$ converges, then $\sum\limits_{k=0}^{\infty} a_k(x-c)^k$ converges absolutely for all $x \in (c-|d|, c+|d|)$.

Corollary 9.82

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists $R > 0$ such that the series converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$.
3. The series converges absolutely for all x.

Definition 9.83: Radius of Convergence and Interval of Convergence

Let a power series centered at c be given. If the power series converges only at c, we say that the radius of convergence of the power series is 0. If the power series converges for $|x-c| < R$ but diverges for $|x-c| > R$, we say that the radius of convergence of the power series is R. If the power series converges for all x, we say that the radius of converges of the power series is ∞. The set of all values of x for which the power series converges is called the interval of convergence of the power series.

Example 9.88. Consider the power series $\sum\limits_{k=1}^{\infty} \frac{x^k}{k}$. Note that for each $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{n+1}\right|}{\left|\frac{x^n}{n}\right|} = \lim_{n\to\infty} \frac{n|x|}{n+1} = |x|\,;$$

thus the ratio test implies that the power series $\sum\limits_{k=1}^{\infty} \frac{x^k}{k}$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. Therefore, the radius of convergence is 1.

To see the interval of convergence, we note that $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges since it is a p-series with $p = 1$, and $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges since it is an alternating series. Therefore, the interval of convergence of the power series is $[-1, 1)$.

Similarly, the power series $\sum\limits_{k=1}^{\infty} \frac{(-1)^k x^k}{k}$ has interval of convergence $(-1, 1]$.

Example 9.89. Consider the power series $\sum\limits_{k=1}^{\infty} \frac{x^k}{k^2}$. Note that for each $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{(n+1)^2}\right|}{\left|\frac{x^n}{n^2}\right|} = \lim_{n\to\infty} \frac{n^2|x|}{(n+1)^2} = |x|\,;$$

thus the ratio test implies that the power series $\sum\limits_{k=1}^{\infty} \frac{x^k}{k^2}$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. Therefore, the radius of convergence is 1.

To see the interval of convergence, we note that $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ converges since it is a p-series with $p = 2$, and $\sum\limits_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ also converges since it converges absolutely (or because of Abel's test). Therefore, the interval of convergence of the power series is $[-1, 1]$.

Remark 9.90. Even though the examples above all has radius of convergence 1, it is not necessary that the radius of convergence of a power series is always 1. For example, the power series $\sum\limits_{k=1}^{\infty} \frac{x^k}{2^k k}$ is obtained by replacing x by $\frac{x}{2}$ in Example 9.88; thus

$$\sum_{k=1}^{\infty} \frac{x^k}{2^k k} \text{ converges for } \frac{x}{2} \in [-1, 1)$$

or equivalent, the interval of convergence of $\sum\limits_{k=1}^{\infty} \frac{x^k}{2^k k}$ is $[-2, 2)$; thus the radius of convergence of this power series is 2.

Example 9.91. The radius of convergence of the power series $\sum\limits_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ is ∞ since for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}x^{2(n+1)+1}}{[2(n+1)+1]!}\right|}{\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right|} = \lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\right|}{\left|\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right|} = \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} = 0\,.$$

- **Differentiation and Integration of Power Series**

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers and $c \in \mathbb{R}$. If the power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ converges in an interval $(c-r, c+r)$, we can ask ourselves whether the function $f : (c-r, c+r)$ defined by $f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k$ is differentiable or not. We note that even though the power series is an infinite sum of differentiable functions (in fact, monomials) it is not clear if the limiting process $\dfrac{d}{dx}$ commutes with $\sum_{k=0}^{\infty}$ since

$$\lim_{n\to\infty}\lim_{h\to 0} nh^2 = 0 \qquad \text{but} \qquad \lim_{h\to 0}\lim_{n\to\infty} nh^2 = \infty\,.$$

Theorem 9.92: Properties of Functions Defined by Power Series

Suppose that the function f defined by power series

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

has a radius of convergence of $R > 0$. Then

1. f is differentiable on $(c-R, c+R)$ and

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots\,.$$

2. an anti-derivative of f on $(c-R, c+R)$ is given by

$$\int f(x)\,dx = C + \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-c)^{k+1} = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \cdots\,.$$

The radius of convergence of the power series obtained by differentiating or integrating a power series term by term is the same as the original power series.

Remark 9.93. Theorem 9.92 states that, in many ways, a function defined by a power series behaves like a polynomial; that is, the derivative (or anti-derivative) of a power series can be obtained by term-by-term differentiation (or integration). However, it is not true for general functions defined by series of the form $\sum_{k=0}^{\infty} b_k(x)$. For example, we have talked about (but did not prove) the series $\sum_{k=1}^{\infty} \dfrac{\sin kx}{k}$ which is the same as $\dfrac{\pi - x}{2}$ on $(0, 2\pi)$; that is,

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \frac{\pi - x}{2} \qquad \forall\, x \in (0, 2\pi)\,.$$

Then

$$-\frac{1}{2} = \frac{d}{dx}\sum_{k=1}^{\infty}\frac{\sin kx}{k} \qquad \forall\, x \in (0, 2\pi)$$

but

$$\frac{d}{dx}\sum_{k=1}^{\infty}\frac{\sin kx}{k} \neq \sum_{k=1}^{\infty}\frac{d}{dx}\frac{\sin kx}{k} = \sum_{k=1}^{\infty}\cos kx \qquad \forall\, x \in (0, 2\pi)$$

since the series $\sum\limits_{k=1}^{\infty}\cos kx$ does not converges for all $x \in (0, 2\pi)$.

Example 9.94. Consider the function f defined by power series

$$f(x) = \sum_{k=1}^{\infty}\frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \qquad \forall\, x \in [-1, 1)\,.$$

Then the function

$$g(x) = \sum_{k=1}^{\infty}x^{k-1} = \sum_{k=0}^{\infty}x^k = 1 + x + x^2 + \cdots\,,$$

obtained by term-by-term differentiation, converges for $x \in (-1, 1)$, and the function

$$h(x) = \sum_{k=1}^{\infty}\frac{x^{k+1}}{k(k+1)} = \sum_{k=2}^{\infty}\frac{x^k}{k(k-1)} = \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{12} + \cdots$$

obtained by term-by-term differentiation, converges for $x \in [-1, 1]$.

Corollary 9.95

For a function defined by power series

$$f(x) = \sum_{k=0}^{\infty}a_k(x-c)^k$$

(on a certain interval of convergence), the n-th Taylor polynomial for f at c is the n-th partial sum $\sum\limits_{k=0}^{n}a_k(x-c)^k$ of the power series.

Proof. Let R be the radius of convergence of the power series. By Theorem 9.92,

$$f'(x) = \sum_{k=1}^{\infty}ka_k(x-c)^{k-1} \qquad \forall\, x \in (c-R, c+R)\,,$$

$$f''(x) = \sum_{k=2}^{\infty}k(k-1)a_k(x-c)^{k-2} \qquad \forall\, x \in (c-R, c+R)\,,$$

thus

$$f(c) = a_0\,, \quad f'(c) = a_1\,, \quad f''(c) = 2a_2\,, \quad \cdots \quad f^{(k)}(c) = k!a_k\,.$$

Therefore, $a_k = \dfrac{f^{(k)}(c)}{k!}$ and the n-th Taylor polynomial for f at c is $\sum\limits_{k=0}^{n} a_k(x-c)^k$. □

9.9 Representation of Functions by Power Series

We have shown the following identities:

$$\begin{aligned}
\exp(x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} && \forall\, x \in \mathbb{R}\,,\\
\sin x &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} && \forall\, x \in \mathbb{R}\,,\\
\cos x &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} && \forall\, x \in \mathbb{R}\,,\\
\ln(1+x) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k} && \forall\, x \in (-1,1]\,.
\end{aligned}$$

In this section, we are interested in finding the power series representation (centered at c) of functions of the form

$$f(x) = \frac{1}{b-x}\,.$$

(without differentiating the function). In other words, for a given $c \in \mathbb{R}\backslash\{b\}$ we would like to find $\{a_k\}_{k=0}^{\infty}$ (which usually depends on c) such that $f(x)$ agrees with the power series

$$\sum_{k=0}^{\infty} a_k(x-c)^k$$

on a certain interval of convergence without differentiating f. For example, we know that

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \forall\, x \in (-1,1)\,;$$

thus to "expand the function about $\frac{1}{2}$"; that is, to write the function $y = \dfrac{1}{1-x}$ as a power series centered at $\frac{1}{2}$, we have

$$\frac{1}{1-x} = \frac{1}{\frac{1}{2} - \big(x-\frac{1}{2}\big)} = 2\cdot\frac{1}{1-2\big(x-\frac{1}{2}\big)} = 2\sum_{k=0}^{\infty}\Big[2\big(x-\tfrac{1}{2}\big)\Big]^k \quad \forall\, x \text{ satisfying } 2\big|x-\tfrac{1}{2}\big| < 1\,.$$

In other words, we obtain

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} 2^{k+1}\left(x - \frac{1}{2}\right)^k \qquad \forall\, x \in (0,1)$$

without computing the derivatives of the function $y = \dfrac{1}{1-x}$ at $\dfrac{1}{2}$.