微積分 MA1002-A 上課筆記（精簡版）

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Theorem 5.41: Cauchy Mean Value Theorem

Let $F, G : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $G'(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in (a, b)$ such that

$$\frac{F'(c)}{G'(c)} = \frac{F(b) - F(a)}{G(b) - G(a)} .$$

Definition 9.69

If f has n derivatives at c, then the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

is called the n-th (order) Taylor polynomial for f at c. The n-th Taylor polynomial for f at 0 is also called the n-th (order) Maclaurin polynomial for f.

- The Maclaurin polynomials for some elementary functions:

1. $y = exp(x) = e^x$:

$$P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} .$$

2. $y = \ln(1 + x)$:

$$P_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}}{n}x^n .$$

3. $y = \sin x$:

$$P_{2n-1}(x) = P_{2n}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n}{(2n-1)!}x^{2n-1} .$$

4. $y = \cos x$:

$$P_{2n}(x) = P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} .$$

- **Remainder of Taylor Polynomials**

The difference $R_n(x) \equiv f(x) - P_n(x)$, where P_n is the n-th Taylor polynomial for f (centered at a certain number c) is called the remainder associated with the approximation P_n.

• **Integral form of the remainder**

Suppose that $f : (a,b) \to \mathbb{R}$ is $(n+1)$-times continuously differentiable, and $c, x \in (a,b)$. Then the remainder R_n associated with the n-th Taylor polynomial for f at c is given by

$$R_n(x) = (-1)^n \int_c^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\,dt\,. \tag{9.7.1}$$

Example 9.74. We have shown last time that if $x > 0$, then

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\,.$$

The identity above holds for $x \leqslant 0$, and the proof is left as an exercise.

Example 9.75. Consider the function $f(x) = \cos x$ and its $(2n)$-th Maclaurin polynomial P_{2n} in Example 9.72. If $x > 0$,

$$\begin{aligned}\big|f(x) - P_{2n}(x)\big| &= \big|f(x) - P_{2n+1}(x)\big| \leqslant \left|\int_0^x f^{(2n+2)}(t)\frac{(t-x)^{2n+1}}{(2n+1)!}\,dt\right| \leqslant \int_0^x \frac{(x-t)^{2n+1}}{(2n+1)!}\,dt \\ &= \frac{-(x-t)^{2n+2}}{(2n+2)!}\bigg|_{t=0}^{t=x} = \frac{x^{2n+2}}{(2n+2)!}\,,\end{aligned}$$

while if $x < 0$,

$$\begin{aligned}\big|f(x) - P_{2n}(x)\big| &= \big|f(x) - P_{2n+1}(x)\big| \leqslant \left|\int_0^x f^{(2n+2)}(t)\frac{(t-x)^{2n+1}}{(2n+1)!}\,dt\right| \leqslant \int_x^0 \frac{(t-x)^{2n+1}}{(2n+1)!}\,dt \\ &= \frac{(t-x)^{2n+2}}{(2n+2)!}\bigg|_{t=0}^{t=x} = \frac{(-x)^{2n+2}}{(2n+2)!}\,.\end{aligned}$$

Therefore,

$$\left|\cos x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}x^{2k}\right| \leqslant \frac{|x|^{2n+2}}{(2n+2)!} \qquad \forall\, x \in \mathbb{R}\,. \tag{9.7.2}$$

Similarly,

$$\left|\sin x - \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}x^{2k+1}\right| \leqslant \frac{|x|^{2n+3}}{(2n+3)!} \qquad \forall\, x \in \mathbb{R}\,. \tag{9.7.3}$$

Moreover, by the fact that

$$\lim_{n\to\infty} \frac{\dfrac{|x|^{2(n+1)+2}}{[2(n+1)+2]!}}{\dfrac{|x|^{2n+2}}{(2n+2)!}} = \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+4)} = 0 < 1$$

and

$$\lim_{n\to\infty} \frac{\dfrac{|x|^{2(n+1)+3}}{[2(n+1)+3]!}}{\dfrac{|x|^{2n+3}}{(2n+3)!}} = \lim_{n\to\infty} \frac{x^2}{(2n+4)(2n+5)} = 0 < 1$$

the ratio test implies that $\sum\limits_{k=0}^{\infty} \dfrac{|x|^{2n+2}}{(2n+2)!}$ and $\sum\limits_{k=0}^{\infty} \dfrac{|x|^{2n+3}}{(2n+3)!}$ converge; thus for each $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{|x|^{2n+2}}{(2n+2)!} = \lim_{n\to\infty} \frac{|x|^{2n+3}}{(2n+3)!} = 0\,.$$

Therefore,

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n} + \cdots\,,$$
$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots\,.$$

Using (9.7.2), we conclude that

$$\Big|\cos(0.1) - \sum_{k=0}^{3} \frac{(-1)^k}{(2k)!} (0.1)^{2k}\Big| \leqslant \frac{0.1^8}{8!}\,;$$

thus $\cos(0.1) \approx \sum\limits_{k=0}^{3} \dfrac{(-1)^k}{(2k)!} (0.1)^{2k} \approx 0.995004165$ which is accurate to nine decimal points.

- **Lagrange form of the remainder**

Theorem 9.76: Taylor's Theorem

Let $f : (a,b) \to \mathbb{R}$ be $(n+1)$-times differentiable, and $c \in (a,b)$. Then for each $x \in (a,b)$, there exists ξ between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)\,, \quad (9.7.4)$$

where the Lagrange form of the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}\,.$$

Proof. We first show that if $h : (a, b) \to \mathbb{R}$ is m-times differentiable, and $c \in (a, b)$. Then for all $d \in (a, b)$ and $d \neq c$ there exists ξ between c and d such that

$$\frac{h(d) - \sum\limits_{k=0}^{m} \frac{h^{(k)}(c)}{k!}(d-c)^k}{(d-c)^{m+1}} = \frac{1}{m+1} \frac{h'(\xi) - \sum\limits_{k=0}^{m-1} \frac{(h')^{(k)}(c)}{k!}(\xi-c)^k}{(\xi-c)^m} . \tag{9.7.5}$$

Let $F(x) = h(x) - \sum\limits_{k=0}^{m} \frac{h^{(k)}(c)}{k!}(x-c)^k$ and $G(x) = (x-c)^{m+1}$. Then F, G are continuous on $[c, d]$ (or $[d, c]$) and differentiable on (c, d) (or (d, c)), and $G'(x) \neq 0$ for all $x \neq c$. Therefore, the Cauchy Mean Value Theorem implies that there exists ξ between c and d such that

$$\frac{F(d) - F(c)}{G(d) - G(c)} = \frac{F'(\xi)}{G'(\xi)} ,$$

and (9.7.5) is exactly the explicit form of the equality above.

Now we apply (9.7.5) successfully for $h = f,\ f',\ f'',\ \cdots$ and $f^{(n)}$ and find that

$$\begin{aligned}
\frac{f(d) - \sum\limits_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(d-c)^k}{(d-c)^{n+1}} &= \frac{1}{n+1} \frac{f'(d_1) - \sum\limits_{k=0}^{n-1} \frac{(f')^{(k)}(c)}{k!}(d_1-c)^k}{(d_1-c)^n} \\
&= \frac{1}{n+1} \cdot \frac{1}{n} \frac{f''(d_2) - \sum\limits_{k=0}^{n-2} \frac{(f'')^{(k)}(c)}{k!}(d_2-c)^k}{(d_2-c)^{n-1}} \\
&= \cdots\cdots \\
&= \frac{1}{(n+1)n(n-1)\cdots 3} \frac{f^{(n-1)}(d_{n-1}) - \sum\limits_{k=0}^{1} \frac{(f^{(n-1)})^{(k)}(c)}{k!}(d_{n-1}-c)^k}{(d_{n-1}-c)^2} \\
&= \frac{1}{(n+1)!} \frac{f^{(n)}(d_n) - f^{(n)}(c)}{d_n - c} = \frac{1}{(n+1)!} f^{(n+1)}(\xi)
\end{aligned}$$

for some $c < \xi < d_n < d_{n-1} < \cdots < d_1 < d$ (or $d < d_1 < d_2 < \cdots < d_n < \xi < c$); thus

$$f(d) - \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(d-c)^k = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(d-c)^{n+1} .$$

(9.7.4) then follows from the equality above since $d \in (a, b)$ is given arbitrary. ▫

Example 9.77. In Example 9.71 we compute the Taylor polynomial P_n for the function $y = \ln(1 + x)$. Note that the Taylor Theorem implies that for all $x > -1$,

$$\ln(1 + x) = P_n(x) + R_n(x) ,$$

where

$$R_n(x) = \frac{1}{(n+1)!}\Big(\frac{d^{n+1}}{dx^{n+1}}\Big|_{x=\xi} \ln(1+x)\Big)x^{n+1} = \frac{(-1)^n}{n+1}(1+\xi)^{-n-1}x^{n+1}$$

for some ξ between 0 and x.

1. If $-1 < x < 0$, then $R_n(x) = \dfrac{-1}{n+1}\Big(\dfrac{-x}{1+\xi}\Big)^{n+1} < 0$; thus

$$\ln(1+x) \leqslant x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n}{n}x^n \qquad \forall\, x \in (-1,0) \text{ and } n \in \mathbb{N}\,.$$

2. If $x > 0$, then

   (a) $R_n(x) < 0$ if n is odd; thus

$$\ln(1+x) \leqslant x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{1}{2k+1}x^{2k+1} \qquad \forall\, x > 0 \text{ and } k \in \mathbb{N}\,.$$

   (b) $R_n(x) > 0$ if n is even; thus

$$\ln(1+x) \geqslant x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{-1}{2k}x^{2k} \qquad \forall\, x > 0 \text{ and } k \in \mathbb{N}\,.$$