微積分 MA1002-A 上課筆記（精簡版）

2019.03.07.

Ching-hsiao Arthur Cheng 鄭經斅

Definition 9.1: Sequence

A ***sequence*** of real numbers (or simply a real sequence) is a function $f : \mathbb{N} \to \mathbb{R}$. We usually use f_n to denote $f(n)$, the n-th term of a sequence $f : \mathbb{N} \to \mathbb{R}$, and this sequence is usually denoted by $\{f_n\}_{n=1}^{\infty}$ or simply $\{f_n\}$.

Definition 9.5

A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is said to ***converge to*** L if for every $\varepsilon > 0$, there exists $N > 0$ such that $|a_n - L| < \varepsilon$ whenever $n \geqslant N$. Such an L (must be a real number and) is called a ***limit*** of the sequence. If $\{a_n\}_{n=1}^{\infty}$ converges to L, we write $a_n \to x$ as $n \to \infty$.

A sequence of real number $\{a_n\}_{n=1}^{\infty}$ is said to be ***convergent*** if there exists $L \in \mathbb{R}$ such that $\{a_n\}_{n=1}^{\infty}$ converges to L. If no such L exists we say that $\{a_n\}_{n=1}^{\infty}$ ***does not converge*** or simply ***diverges***.

Proposition 9.6

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, and $a_n \to a$ and $a_n \to b$ as $n \to \infty$, then $a = b$. (若收斂則極限唯一).

• **Notation**: Since the limit of a convergent sequence is unique, we use $\lim\limits_{n\to\infty} a_n$ to denote this unique limit of a convergent sequence $\{a_n\}_{n=1}^{\infty}$.

Theorem 9.7

Let L be a real number, and $f : [1, \infty) \to \mathbb{R}$ be a function of a real variable such that $\lim\limits_{x\to\infty} f(x) = L$. If $\{a_n\}_{n=1}^{\infty}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then $\lim\limits_{n\to\infty} a_n = L$.

Theorem 9.11

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = K$. Then

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = L \pm K$.
2. $\lim\limits_{n\to\infty} (a_n b_n) = LK$. In particular, $\lim\limits_{n\to\infty} (ca_n) = cL$ if c is a real number.
3. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{K}$ if $K \neq 0$.

Theorem 9.12: Squeeze Theorem

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $a_n \leqslant c_n \leqslant b_n$ for all $n \geqslant N$. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then $\lim_{n\to\infty} c_n = L$.

Theorem 9.13: Absolute Value Theorem

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Definition 9.14: Monotonicity of Sequences

A sequence $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ is said to be

1. (**monotone**) **increasing** if $a_{n+1} \geqslant a_n$ for all $n \in \mathbb{N}$;
2. (**monotone**) **decreasing** if $a_{n+1} \leqslant a_n$ for all $n \in \mathbb{N}$;
3. ***monotone*** if $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence or a decreasing sequence.

Definition 9.16: Boundedness of Sequences

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

1. $\{a_n\}_{n=1}^{\infty}$ is said to be ***bounded***（有界的）if there exists $M \in \mathbb{R}$ such that $|a_n| \leqslant M$ for all $n \in \mathbb{N}$.
2. $\{a_n\}_{n=1}^{\infty}$ is said to be ***bounded from above***（有上界）if there exists $B \in \mathbb{R}$, called an ***upper bound*** of the sequence, such that $a_n \leqslant B$ for all $n \in \mathbb{N}$. Such a number B is called an upper bound of the sequence.
3. $\{a_n\}_{n=1}^{\infty}$ is said to be ***bounded from below***（有下界）if there exists $A \in \mathbb{R}$, called a ***lower bound*** of the sequence, such that $A \leqslant a_n$ for all $n \in \mathbb{N}$. Such a number A is called a lower bound of the sequence.

Proposition 9.18

A convergent sequence of real numbers is bounded（數列收斂必有界）.

- **Completeness of Real Numbers**:

One important property of the real numbers is that they are ***complete***. The completeness axiom for real numbers states that "every bounded sequence of real numbers has a ***least upper bound*** and a ***greatest lower bound***"; that is, if $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers, then there exists an upper bound M and a lower bound m of $\{a_n\}_{n=1}^{\infty}$ such that there is no smaller upper bound nor greater lower bound of $\{a_n\}_{n=1}^{\infty}$.

Theorem 9.20: Monotone Sequence Property (MSP)

Let $\{a_n\}_{n=1}^{\infty}$ be a monotone sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ converges if and only if $\{a_n\}_{n=1}^{\infty}$ is bounded.

Proof. It suffices to show the "$\Leftarrow$" direction.

Without loss of generality, we can assume that $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded. By the completeness of real numbers, there exists a least upper bound M for the sequence $\{a_n\}_{n=1}^{\infty}$.

Let $\varepsilon > 0$ be given. Since M is the least upper bound for $\{a_n\}_{n=1}^{\infty}$, $M - \varepsilon$ is not an upper bound; thus there exists $N \in \mathbb{N}$ such that $a_N > M - \varepsilon$. Since $\{a_n\}_{n=1}^{\infty}$ is increasing, $a_n \geqslant a_N$ for all $n \geqslant N$. Therefore,

$$M - \varepsilon < a_n \leqslant M \qquad \forall\, n \geqslant N$$

which implies that

$$|a_n - M| < \varepsilon \qquad \forall\, n \geqslant N\,.$$

The statement above shows that $\{a_n\}_{n=1}^{\infty}$ converges to M. □

Remark 9.21. A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is called a ***Cauchy sequence*** if for every $\varepsilon > 0$ there exists $N > 0$ such that

$$|a_n - a_m| < \varepsilon \qquad \text{whenever} \quad n, m \geqslant N\,.$$

A convergent sequence must be a Cauchy sequence. Moreover, the completeness of real numbers is equivalent to that "every Cauchy sequence of real number converges".

9.2 Series and Convergence

An infinite series is the "sum" of an infinite sequence. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, then

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \cdots + a_n + \cdots$$

is an infinite series (or simply series). The numbers a_1, a_2, a_3, $\cdots$ are called the terms of the series. For convenience, the sum could begin the index at $n = 0$ or some other integer.

Definition 9.22

The series $\sum_{k=1}^{\infty} a_k$ is said to converge to S if the sequence of the partial sum, denoted by $\{S_n\}_{n=1}^{\infty}$ and defined by

$$S_n \equiv \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n,$$

converges to S. S_n is called the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$.

When the series converges, we write $S = \sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} a_k$ is said to be convergent. If $\{S_n\}_{n=1}^{\infty}$ diverges, the series is said to be divergent or diverge. If $\lim_{n\to\infty} S_n = \infty$ (or $-\infty$), the series is said to diverge to ∞ (or $-\infty$).

Example 9.23. The n-th partial sum of the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ is

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \Big(\frac{1}{k} - \frac{1}{k+1}\Big) = \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \cdots + \Big(\frac{1}{n} - \frac{1}{n+1}\Big) \\ &= 1 - \frac{1}{n+1}\,; \end{aligned}$$

thus the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges to 1, and we write $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$.

Example 9.24. The n-th partial sum of the series $\sum_{k=1}^{\infty} \frac{2}{4k^2-1}$ is

$$\begin{aligned} \sum_{k=1}^{n} \frac{2}{4k^2-1} &= \sum_{k=1}^{n} \frac{2}{(2k-1)(2k+1)} = \sum_{k=1}^{n} \Big(\frac{1}{2k-1} - \frac{1}{2k+1}\Big) \\ &= \Big(1 - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{1}{5}\Big) + \cdots + \Big(\frac{1}{2n-1} - \frac{1}{2n+1}\Big) = 1 - \frac{1}{2n+1}\,; \end{aligned}$$

thus the series $\sum_{k=1}^{\infty} \frac{2}{4k^2-1}$ converges to 1, and we write $\sum_{k=1}^{\infty} \frac{2}{4k^2-1} = 1$.

The series in the previous two examples are series of the form

$$\sum_{k=1}^{n} (b_k - b_{k+1}) = (b_1 - b_2) + (b_2 - b_3) + \cdots + (b_n - b_{n+1}) + \cdots ,$$

and are called telescoping series. A telescoping series converges if and only if $\lim_{n\to\infty} b_n$ converges.

Example 9.25. The series $\sum_{k=1}^{\infty} r^k$, where r is a real number, is called a geometric series (with ratio r). Note that the n-th partial sum of the series is

$$S_n = \sum_{k=1}^{n} r^k = 1 + r + r^2 + \cdots + r^n = \begin{cases} \dfrac{1-r^{n+1}}{1-r} & \text{if } r \neq 1\,, \\ n+1 & \text{if } r = 1\,. \end{cases}$$

Therefore, the geometric series converges if and only if the common ratio r satisfies $|r| < 1$.

Theorem 9.26

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty}$ be convergent series, and c is a real number. Then

1. $\sum_{k=1}^{\infty} ca_k = c \sum_{k=1}^{\infty} a_k$.
2. $\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$.
3. $\sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$.

Theorem 9.27: Cauchy Criteria

A series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\varepsilon > 0$, there exists $N > 0$ such that

$$\Big|\sum_{k=n}^{n+p} a_k\Big| < \varepsilon \qquad \text{whenever} \quad n \geqslant N, p \geqslant 0\,.$$

Proof. Let S_n be the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$. Then by Remark 9.21,

$$\begin{aligned} \sum_{k=1}^{\infty} a_k \text{ converges} &\Leftrightarrow \{S_n\}_{n=1}^{\infty} \text{ is a convergent sequence} \\ &\Leftrightarrow \{S_n\}_{n=1}^{\infty} \text{ is a Cauchy sequence} \\ &\Leftrightarrow \text{for every } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that} \\ &\qquad\qquad |S_n - S_m| < \varepsilon \text{ whenever } n, m \geqslant N \\ &\Leftrightarrow \text{for every } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that} \\ &\qquad\qquad |a_n + a_{n+1} + \cdots + a_{n+p}| < \varepsilon \text{ whenever } n \geqslant N \text{ and } p \geqslant 0. \end{aligned}$$

□

Corollary 9.28

If the series $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.

Remark 9.29. It is not true that $\lim_{n\to\infty} a_n = 0$ implies the convergence of $\sum_{k=1}^{\infty} a_k$. For example, we have shown in Example 8.47 that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to ∞ while we know that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Corollary 9.30: n-th term test for divergence

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then the series $\sum_{k=1}^{\infty} a_k$ diverges.