微積分 MA1002-A 上課筆記（精簡版）

2019.02.21.

Ching-hsiao Arthur Cheng 鄭經斅

Definition 8.31: Improper Integrals with Infinite Integration Limits

1. If f is Riemann integrable on the interval $[a, b]$ for all $a < b$, then

$$\int_a^\infty f(x)\,dx \equiv \lim_{b\to\infty} \int_a^b f(x)\,dx\,.$$

2. If f is Riemann integrable on the interval $[a, b]$ for all $a < b$, then

$$\int_{-\infty}^b f(x)\,dx \equiv \lim_{a\to-\infty} \int_a^b f(x)\,dx\,.$$

3. If f is Riemann integrable on the interval $[a, b]$ for all $a < b$, then

$$\int_{-\infty}^\infty f(x)\,dx \equiv \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx\,,$$

   where c is any real number.

Definition 8.38: Improper integrals with Infinite Discontinuities

1. If f is Riemann integrable on $[a, c]$ for all $a < c < b$, and f has an infinite discontinuity at b; that is, $\lim_{x\to b^-} f(x) = \infty$ or $-\infty$, then

$$\int_a^b f(x)\,dx \equiv \lim_{c\to b^-} \int_a^c f(x)\,dx\,.$$

2. If f is Riemann integrable on $[c, b]$ for all $a < c < b$, and f has an infinite discontinuity at a; that is, $\lim_{x\to a^+} f(x) = \infty$ or $-\infty$, then

$$\int_a^b f(x)\,dx \equiv \lim_{c\to a^+} \int_c^b f(x)\,dx\,.$$

3. Suppose that $a < c < b$. If f is Riemann integrable on $[a, c-\epsilon]$ and $[c+\epsilon, b]$ for all $0 < \epsilon \ll 1$, and f has an infinite discontinuity at c; that is $\lim_{x\to c^+} f(x) = \infty$ or $-\infty$ and $\lim_{x\to c^-} f(x) = \infty$ or $-\infty$, then

$$\int_a^b f(x)\,dx \equiv \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

The convergence and divergence of the improper integrals with infinite discontinuities are similar to the statements in Definition 8.31.

Definition 8.45

Let $\int_a^b f(x)\,dx$, where a, b could be infinitely, be an improper integral.

1. The improper integral $\int_a^b f(x)\,dx$ is said to be absolutely convergent or converge absolutely if $\int_a^b \big|f(x)\big|\,dx$ converges.
2. The improper integral $\int_a^b f(x)\,dx$ is said to be conditionally convergent or converge conditionally if $\int_a^b f(x)\,dx$ converges but $\int_a^b \big|f(x)\big|\,dx$ diverges (to ∞).

Remark 8.46. Even though it is not required in the definition that an absolutely convergent improper integral has to converge, it is in fact true an absolutely convergent improper integral converges.

Theorem 8.48: A special type of improper integral

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} & \text{if } p > 1\,, \\ \text{diverges to } \infty & \text{if } p \leqslant 1\,. \end{cases}$$

• **Comparison Test for Improper Integrals**

In the last part of this section, we consider some criteria which can be used to judge if an improper integral converges or diverges, without evaluating the exact value of the improper integral.

Theorem 8.49

Let f and g be continuous functions and $0 \leqslant g(x) \leqslant f(x)$ on the interval $[a, \infty)$.

1. If the improper integral $\int_a^\infty f(x)\,dx$ converges, then the improper integral $\int_a^\infty g(x)\,dx$ converges.
2. If the improper integral $\int_a^\infty g(x)\,dx$ diverges to ∞, then the improper integral $\int_a^\infty f(x)\,dx$ diverges.

The result also holds for improper integrals given by other two cases in Definition 8.31 and the case with infinite discontinuities.

Proof. For $b > a$, define $G(b) = \int_a^b g(x)\,dx$ and $F(b) = \int_a^b f(x)\,dx$. By the Fundamental Theorem of Calculus, $F, G : [a, \infty) \to \mathbb{R}$ is differentiable (hence continuous). Since $0 \leqslant g(x) \leqslant f(x)$ on $[a, \infty)$, for all $b > a$ we have $0 \leqslant G(b) \leqslant F(b)$, and F, G are monotone increasing.

1. If the improper integral $\int_a^\infty f(x)\,dx$ converges, the $\lim\limits_{b\to\infty} F(b) = M$ exists. Since F is monotone increasing, $F(b) \leqslant M$ for all $b > a$; thus $G(b) \leqslant M$ for all $b > a$. By the monotonicity of G, $\lim\limits_{b\to\infty} G(b)$ exists.
2. If the improper integral $\int_a^\infty g(x)\,dx$ diverges to ∞, $\lim\limits_{b\to\infty} G(b) = \infty$; thus the fact that $G(b) \leqslant F(b)$ implies that $\lim\limits_{b\to\infty} F(b) = \infty$. □

Example 8.50. Consider the improper integral $\int_1^\infty e^{-x^2}\,dx$. Note that $e^{-x^2} \leqslant e^{-x}$ for all $x \in [1, \infty)$. Since

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} (-e^{-x})\Big|_{x=1}^{x=b} = \lim_{b\to\infty} (e^{-b} - e^{-1}) = -e^{-1}\,,$$

by Theorem 8.49 we find that the improper integral $\int_1^\infty e^{-x^2}\,dx$ converges.

Example 8.51. Consider the improper integral $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$. Note that $\frac{\sin^2 x}{x^2} \leqslant \frac{1}{x^2}$ for all $x \in [1, \infty)$. Since

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \big(-\frac{1}{x}\big)\Big|_{x=1}^{x=b} = \lim_{b\to\infty} \big(\frac{1}{b} - 1\big) = -1\,,$$

by Theorem 8.49 we find that the improper integral $\int_1^\infty e^{-x^2}\,dx$ converges.

Example 8.52 (The Gamma Function)**.** The Gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt\,.$$

We note that for each $x \in \mathbb{R}$, the integrand $f(t) = t^{x-1}e^{-t}$ is positive on $[0, \infty)$.

1. If $x \geqslant 1$, the function $y = t^{x-1}e^{-\frac{t}{2}}$ is differentiable on $[0, \infty)$ and has a maximum at the point $t = 2(x-1)$. Therefore,

$$0 \leqslant f(t) \leqslant 2^{x-1}(x-1)^{x-1}e^{-\frac{t}{2}} \qquad \forall\, t \geqslant 0\,.$$

By the fact that

$$\int_0^\infty e^{-\frac{t}{2}}\,dt = \lim_{b\to\infty}\int_0^b e^{-\frac{t}{2}}\,dt = \lim_{b\to\infty}\big(-2e^{-\frac{t}{2}}\big)\Big|_{t=0}^{t=b} = \lim_{b\to\infty}\big(2-2e^{-\frac{b}{2}}\big) = 2,$$

we find that the improper integral $\displaystyle\int_0^\infty t^{x-1}e^{-t}\,dt$ converges.

2. If $0 < x < 1$, the function f has an infinite discontinuity at 0. Therefore,

$$\int_0^\infty t^{x-1}e^{-t}\,dt = \int_0^1 t^{x-1}e^{-t}\,dt + \int_1^\infty t^{x-1}e^{-t}\,dt\,.$$

Again, the function $y = t^{x-1}e^{-\frac{t}{2}}$ is bounded from above by $2^{x-1}(x-1)^{x-1}$; thus the same reason as above show that the improper integral $\displaystyle\int_1^\infty t^{x-1}e^{-t}\,dt$ converges.

On the other hand, note that $f(t) \leqslant t^{x-1}$ for all $t \in [0,1]$. By the fact that

$$\int_0^1 t^{x-1}\,dt = \lim_{a\to 0^+}\int_a^1 t^{x-1}\,dx = \lim_{a\to 0^+}\frac{t^x}{x}\Big|_{t=a}^{t=1} = \lim_{a\to 0^+}\frac{1-a^x}{x} = \frac{1}{x}\,,$$

we find that the improper integral $\displaystyle\int_0^1 t^{x-1}e^{-t}\,dt$ converges. Therefore, the improper integral $\displaystyle\int_0^\infty t^{x-1}e^{-t}\,dt$ converges.

3. If $x \leqslant 0$, then $t^{x-1}e^{-t} \geqslant t^{x-1}e^{-1}$ for all $t \in [0,1]$. By the fact that

$$\int_0^1 t^{x-1}e^{-1}\,dt = \lim_{a\to 0^+}\int_a^1 t^{x-1}e^{-1}\,dt = \infty\,,$$

Theorem 8.49 implies that the improper integral $\displaystyle\int_0^1 t^{x-1}e^{-t}\,dt$ diverges to ∞. This implies that the improper integral $\displaystyle\int_0^\infty t^{x-1}e^{-t}\,dt$ diverges to ∞ as well.