微積分 MA1002-A 上課筆記（精簡版）

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Ching-hsiao Arthur Cheng 鄭經斅

Definition 8.31: Improper Integrals with Infinite Integration Limits

1. If f is Riemann integrable on the interval $[a, b]$ for all $a < b$, then
$$\int_a^\infty f(x)\,dx \equiv \lim_{b\to\infty} \int_a^b f(x)\,dx\,.$$
2. If f is Riemann integrable on the interval $[a, b]$ for all $a < b$, then
$$\int_{-\infty}^b f(x)\,dx \equiv \lim_{a\to-\infty} \int_a^b f(x)\,dx\,.$$
3. If f is Riemann integrable on the interval $[a, b]$ for all $a < b$, then
$$\int_{-\infty}^\infty f(x)\,dx \equiv \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx\,,$$
where c is any real number.

Theorem 8.37

1. If f is Riemann integrable on the interval $[a, b]$ for all $a < b$, then
$$\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx \qquad \forall\, a < c\,,$$
provided that the improper integrals on both sides converge or diverge to ∞ (or $-\infty$).
2. If f is Riemann integrable on the interval $[a, b]$ for all $a < b$, then
$$\int_{-\infty}^b f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^b f(x)\,dx \qquad \forall\, c < b\,,$$
provided that the improper integrals on both sides converge or diverge to ∞ (or $-\infty$).
3. If f is Riemann integrable on the interval $[a, b]$ for all $a < b$ and $\int_{-\infty}^\infty f(x)\,dx$ converges or diverges to ∞ (or $-\infty$), then
$$\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx = \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx \qquad \forall\, a, b \in \mathbb{R}\,.$$

Proof. We only prove 1 and 3, for the proof of 2 is similar to the proof of 1.

1. By the properties of the definite integrals, for $a < c$ we have
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,;$$
thus
$$\begin{aligned}\int_a^\infty f(x)\,dx &= \lim_{b\to\infty}\int_a^b f(x)\,dx = \lim_{b\to\infty}\Big[\int_a^c f(x)\,dx + \int_c^b f(x)\,dx\Big]\\ &= \int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx\,.\end{aligned}$$

3. If $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ converges or diverges to ∞ (or $-\infty$), then both improper integrals $\displaystyle\int_c^\infty f(x)\,dx$ and $\displaystyle\int_{-\infty}^c f(x)\,dx$ converge or diverge to ∞ (or $-\infty$). Therefore,
$$\begin{aligned}\int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx &= \int_{-\infty}^a f(x)\,dx + \int_a^b f(x)\,dx + \int_b^\infty f(x)\,dx\\ &= \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx\,.\end{aligned}$$
□

Definition 8.38: Improper integrals with Infinite Discontinuities

1. If f is Riemann integrable on $[a,c]$ for all $a < c < b$, and f has an infinite discontinuity at b; that is, $\displaystyle\lim_{x\to b^-} f(x) = \infty$ or $-\infty$, then
$$\int_a^b f(x)\,dx \equiv \lim_{c\to b^-}\int_a^c f(x)\,dx\,.$$
2. If f is Riemann integrable on $[c,b]$ for all $a < c < b$, and f has an infinite discontinuity at a; that is, $\displaystyle\lim_{x\to a^+} f(x) = \infty$ or $-\infty$, then
$$\int_a^b f(x)\,dx \equiv \lim_{c\to a^+}\int_c^b f(x)\,dx\,.$$
3. Suppose that $a < c < b$. If f is Riemann integrable on $[a, c-\epsilon]$ and $[c+\epsilon, b]$ for all $0 < \epsilon \ll 1$, and f has an infinite discontinuity at c; that is $\displaystyle\lim_{x\to c^+} f(x) = \infty$ or $-\infty$ and $\displaystyle\lim_{x\to c^-} f(x) = \infty$ or $-\infty$, then
$$\int_a^b f(x)\,dx \equiv \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

The convergence and divergence of the improper integrals with infinite discontinuities are similar to the statements in Definition 8.31.

Example 8.39. Evaluate $\displaystyle\int_0^1 x^{-\frac{1}{3}}\,dx$.

We observe that the integrand has an infinite discontinuity at 0. Therefore,

$$\int_0^1 x^{-\frac{1}{3}}\,dx = \lim_{a\to 0^+}\int_a^1 x^{-\frac{1}{3}}\,dx = \lim_{a\to 0^+}\frac{3}{2}x^{\frac{2}{3}}\Big|_{x=a}^{x=1} = \lim_{a\to 0^+}\frac{3}{2}(1-a^{\frac{2}{3}}) = \frac{3}{2}\,.$$

Example 8.40. Evaluate $\displaystyle\int_0^2 x^{-3}\,dx$.

We observe that the integrand has an infinite discontinuity at 0. Therefore,

$$\int_0^2 x^{-3}\,dx = \lim_{a\to 0^+}\int_a^2 x^{-3}\,dx = \lim_{a\to 0^+}\frac{-x^{-2}}{2}\Big|_{x=a}^{x=2} = \lim_{a\to 0^+}\Big(-\frac{1}{8}+\frac{1}{2a^2}\Big) = \infty\,;$$

thus the improper integral $\displaystyle\int_0^2 x^{-3}\,dx$ diverges to ∞.

Example 8.41. Evaluate $\displaystyle\int_{-1}^2 x^{-3}\,dx$.

Since the integrand has an infinite discontinuity at 0,

$$\int_{-1}^2 x^{-3}\,dx = \int_{-1}^0 x^{-3}\,dx + \int_0^2 x^{-3}\,dx\,.$$

We have shown in previous example that the second integral on the right-hand side diverges to ∞. Similarly, the first integral on the right-hand side diverges to $-\infty$ since

$$\int_{-1}^0 x^{-3}\,dx = \lim_{b\to 0^-}\int_{-1}^b x^{-3}\,dx = \lim_{b\to 0^-}\frac{-x^{-2}}{2}\Big|_{x=-1}^{x=b} = \lim_{b\to 0^-}\Big(-\frac{1}{2b^2}+\frac{1}{2}\Big) = -\infty\,;$$

thus the improper integral $\displaystyle\int_{-1}^2 x^{-3}\,dx$ diverges (but not diverges to ∞ or $-\infty$).

Remark 8.42. Even though $y = -\dfrac{x^{-2}}{2}$ is an anti-derivative of the function $y = x^{-3}$, you cannot apply the "Fundamental Theorem of Calculus" to conclude that

$$\int_{-1}^2 x^{-3}\,dx = \frac{x^{-2}}{-2}\Big|_{x=-1}^{x=2} = -\frac{1}{8}+\frac{1}{2} = \frac{3}{8}$$

since $y = x^{-3}$ is not Riemann integrable on $[-1,2]$.

Similar to Theorem 8.37, we also have the following

Theorem 8.43

If f is Riemann integrable on $[a, c]$ for all $a < c < b$, and f has an infinite discontinuity at a or b, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \qquad \forall\, a < c < b\,,$$

provided that the improper integrals on both sides converge or diverge to ∞ (or $-\infty$).

We can also consider improper integral $\displaystyle\int_a^b f(x)\,dx$ in which $a = -\infty$ or $b = \infty$, and f has an infinite discontinuity at c for $a < c < b$. In this case, we define

$$\int_a^{\infty} f(x)\,dx = \int_a^d f(x)\,dx + \int_d^{\infty} f(x)\,dx \qquad \forall\, d > c\,,$$
$$\int_{-\infty}^b f(x)\,dx = \int_{-\infty}^d f(x)\,dx + \int_d^b f(x)\,dx \qquad \forall\, d < c\,,$$

and etc. In other words, when the integrand and the domain of integration are unbounded, we divide the integral into improper integrals of one type and compute those integrals separately, pretending that the summing rule

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \cdots + \int_{c_{n-1}}^{c_n} f(x)\,dx + \int_{c_n}^b f(x)\,dx$$

also holds for improper integrals.

Example 8.44. Evaluate $\displaystyle\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$.

We observe that the integrand has an infinite discontinuity at 0, and the domain of integration is unbounded. Therefore,

$$\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)} = \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^{\infty} \frac{dx}{\sqrt{x}(x+1)}\,.$$

By the substitution $u = \sqrt{x}$, $du = \dfrac{dx}{2\sqrt{x}}$; thus

$$\int \frac{dx}{\sqrt{x}(x+1)} = \int \frac{2du}{u^2+1} = 2\arctan u + C = 2\arctan\sqrt{x} + C\,.$$

Therefore,

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{x}(x+1)} &= \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}(x+1)} = \lim_{a\to 0^+} 2\arctan\sqrt{x}\Big|_{x=a}^{x=1} \\ &= \lim_{a\to 0^+} \Big(2\cdot\frac{\pi}{4} - 2\arctan\sqrt{a}\Big) = \frac{\pi}{2}\end{aligned}$$

and

$$\int_1^\infty \frac{dx}{\sqrt{x}(x+1)} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}(x+1)} = \lim_{b\to\infty} 2\arctan\sqrt{x}\Big|_{x=1}^{x=b}$$
$$= \lim_{b\to\infty}\Big(2\arctan\sqrt{b} - 2\cdot\frac{\pi}{4}\Big) = \pi - \frac{\pi}{2} = \frac{\pi}{2}\,.$$

As a consequence,

$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \frac{\pi}{2} + \frac{\pi}{2} = \pi\,.$$

Definition 8.45

Let $\displaystyle\int_a^b f(x)\,dx$, where a, b could be infinitely, be an improper integral.

1. The improper integral $\displaystyle\int_a^b f(x)\,dx$ is said to be absolutely convergent or converge absolutely if $\displaystyle\int_a^b \big|f(x)\big|\,dx$ converges.
2. The improper integral $\displaystyle\int_a^b f(x)\,dx$ is said to be conditionally convergent or converge conditionally if $\displaystyle\int_a^b f(x)\,dx$ converges but $\displaystyle\int_a^b \big|f(x)\big|\,dx$ diverges (to ∞).

Remark 8.46. Even though it is not required in the definition that an absolutely convergent improper integral has to converge, it is in fact true an absolutely convergent improper integral converges.

Example 8.47. The improper integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ is conditionally convergent but not absolutely convergent. To see that the improper integral is not absolutely convergent, we note that if $n \in \mathbb{N}$,

$$\int_0^{2n\pi}\Big|\frac{\sin x}{x}\Big|\,dx = \sum_{k=1}^n \int_{2(k-1)\pi}^{2k\pi}\Big|\frac{\sin x}{x}\Big|\,dx = \sum_{k=1}^n \int_0^{2\pi}\Big|\frac{\sin\big[u+2(k-1)\pi\big]}{u+2(k-1)\pi}\Big|\,du$$
$$= \sum_{k=1}^n \int_0^{2\pi}\frac{|\sin u|}{|u+2(k-1)\pi|}\,du = \sum_{k=1}^n \int_0^{2\pi}\frac{|\sin u|}{2k\pi}\,du \geqslant \frac{2}{\pi}\sum_{k=1}^n \frac{1}{k}\,;$$

thus by the fact that

$$\begin{aligned}\sum_{k=1}^{2^n}\frac{1}{k} &= 1+\frac{1}{2}+\Big(\frac{1}{3}+\frac{1}{4}\Big)+\Big(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\Big)+\cdots+\Big(\frac{1}{2^{n-1}+1}+\frac{1}{2^{n-1}+2}+\cdots+\frac{1}{2^n}\Big)\\ &\geqslant 1+\frac{1}{2}+\Big(\frac{1}{4}+\frac{1}{4}\Big)+\Big(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\Big)+\cdots+\Big(\underbrace{\frac{1}{2^n}+\frac{1}{2^n}+\cdots+\frac{1}{2^n}}_{2^{n-1}\text{ terms}}\Big)\\ &= 1+\underbrace{\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2}}_{(n-1)\text{ terms}}=\frac{n+1}{2}\geqslant\frac{n}{2}\,,\end{aligned}$$

we find that

$$\int_0^{2^{n+1}\pi}\Big|\frac{\sin x}{x}\Big|\,dx\geqslant\frac{2}{\pi}\sum_{k=1}^{2^n}\frac{1}{k}\geqslant\frac{n}{\pi}$$

which approaches ∞ as $n\to\infty$.