微積分 MA1002-A 上課筆記（精簡版）

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Definition 4.7: Riemann Integrals - 黎曼積分

Let $f : [a, b] \to \mathbb{R}$ be a function. f is said to be Riemann integrable on $[a, b]$ if there exists a real number A such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums for the partition $\mathcal{P}$ belongs to the interval $(A - \varepsilon, A + \varepsilon)$. Such a number A (is unique and) is called the Riemann integral of f on $[a, b]$ and is denoted by $\displaystyle\int_{[a,b]} f(x)\,dx$ $\Big($or simply $\displaystyle\int_a^b f(x)\,dx\Big)$.

Theorem 4.31: "The Fundamental Theorem of Calculus"

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function and F be an anti-derivative of f on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Moreover, if in addition f is continuous on $[a, b]$, then $G(x) = \displaystyle\int_a^x f(t)\,dt$ is differentiable on $[a, b]$ and

$$G'(x) = f(x) \qquad \text{for all } x \in [a, b]\,.$$

8.5 Improper Integrals - 瑕積分

Recall that given a non-negative continuous function $f : [a, b] \to \mathbb{R}$, the area of the region enclosed by the graph of f, the x-axis and lines $x = a$, $x = b$ is given by

$$\int_a^b f(x)\,dx\,.$$

What happened when

1. the function under consideration is non-negative and continuous on the whole real line and we would like to know, for example, the area of the region enclosed by the graph of f and the x-axis and is on the right-hand (or left-hand) side of the line $x = c$?

2. the function under consideration blows up at a point $c \in [a, b]$; that is, $\lim\limits_{x \to c^{\pm}} f(x)$ diverges to ∞ or $-\infty$ (so that f is not continuous at c but everywhere else) and we would like to know the area of the region enclosed by the graph of f, the x-axis and lines $x = a$ and $x = b$?

Note that the definition of a definite integral $\int_a^b f(x)\,dx$ requires that the interval $[a,b]$ be finite and f be bounded. Therefore, $\int_a^\infty f(x)\,dx$, $\int_{-\infty}^b f(x)\,dx$ and $\int_a^b f(x)\,dx$ when f is unbounded are meaningless in the sense of Riemann integrals. How do we compute the area of those unbounded regions?

Definition 8.31: Improper Integrals with Infinite Integration Limits

1. If f is Riemann integrable on the interval $[a,b]$ for all $a<b$, then
$$\int_a^\infty f(x)\,dx \equiv \lim_{b\to\infty}\int_a^b f(x)\,dx\,.$$
2. If f is Riemann integrable on the interval $[a,b]$ for all $a<b$, then
$$\int_{-\infty}^b f(x)\,dx \equiv \lim_{a\to-\infty}\int_a^b f(x)\,dx\,.$$
3. If f is Riemann integrable on the interval $[a,b]$ for all $a<b$, then
$$\int_{-\infty}^\infty f(x)\,dx \equiv \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx\,,$$
where c is any real number.

In the first two cases, the improper integral converges when the limit exists. Otherwise, the improper integral diverges. If the limits, as b approaches ∞ (or a approaches $-\infty$), approaches ∞ or $-\infty$, then the improper integral diverges to ∞ or $-\infty$. In the third case, the improper integral on the left converges when both of the improper integrals on the right converges, and diverges when either of the improper integrals on the right diverges. The improper integral on the left diverges to ∞ (or $-\infty$) if it diverges and the improper integrals on the right is $\infty+\infty$, $\infty+C$ or $C+\infty$ (or $(-\infty)+(-\infty)$, $(-\infty)+C$ or $C+(-\infty)$).

Example 8.32. Evaluate $\int_0^\infty e^{-x}\,dx$ and $\int_0^\infty \frac{1}{x^2+1}\,dx$.

Since an anti-derivative of the function $y=e^{-x}$ and $y=\frac{1}{x^2+1}$ is $y=-e^{-x}$ and $y=\arctan x$, the Fundamental Theorem of Calculus implies that

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}(-e^{-x})\Big|_{x=0}^{x=b} = \lim_{b\to\infty}(1-e^{-b}) = 1-\lim_{b\to\infty}e^{-b} = 1$$

and

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\arctan x\Big|_{x=0}^{x=b} = \lim_{b\to\infty}\arctan b = \frac{\pi}{2}\,.$$

Example 8.33. Evaluate $\displaystyle\int_1^\infty (1-x)e^{-x}\,dx$.

Let $u = 1-x$ and $v = -e^{-x}$ (so that $dv = e^{-x}\,dx$). For any real number b, integration by parts implies that

$$\begin{aligned}\int_1^b (1-x)e^{-x}\,dx &= \big[(1-x)(-e^{-x})\big]\Big|_{x=1}^{x=b} - \int_1^b (-e^{-x})(-dx) = -(1-b)e^{-b} - \int_1^b e^{-x}\,dx \\ &= -(1-b)e^{-b} + e^{-x}\Big|_{x=1}^{x=b} = -(1-b)e^{-b} + e^{-b} - e^{-1} = be^{-b} - e^{-1}\,.\end{aligned}$$

Therefore,

$$\int_1^\infty (1-x)e^{-x}\,dx = \lim_{b\to\infty}\int_1^b (1-x)e^{-x}\,dx = \lim_{b\to\infty}(be^{-b} - e^{-1}) = -e^{-1}\,.$$

Example 8.34. Evaluate $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx$.

To evaluate the integral above, we evaluate the two integrals

$$\int_0^\infty \frac{e^x}{1+e^{2x}}\,dx \quad\text{and}\quad \int_{-\infty}^0 \frac{e^x}{1+e^{2x}}\,dx\,.$$

By the substitution of variable $u = e^x$, we find that $du = e^x\,dx$; thus

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1+u^2}\,du = \arctan u + C = \arctan(e^x) + C\,.$$

Therefore,

$$\begin{aligned}\int_0^\infty \frac{e^x}{1+e^{2x}}\,dx &= \lim_{b\to\infty}\int_0^b \frac{e^x}{1+e^{2x}}\,dx = \lim_{b\to\infty}\arctan(e^x)\Big|_{x=0}^{x=b} \\ &= \lim_{b\to\infty}\big[\arctan(e^b) - \arctan 1\big] = \frac{\pi}{4}\end{aligned}$$

and similarly,

$$\begin{aligned}\int_{-\infty}^0 \frac{e^x}{1+e^{2x}}\,dx &= \lim_{a\to-\infty}\int_a^0 \frac{e^x}{1+e^{2x}}\,dx = \lim_{a\to-\infty}\arctan(e^x)\Big|_{x=a}^{x=0} \\ &= \lim_{a\to-\infty}\big[\arctan 1 - \arctan(e^a)\big] = \frac{\pi}{4}\,.\end{aligned}$$

The two integrals above implies that $\displaystyle\int_{-\infty}^\infty \frac{e^x}{1+e^{2x}}\,dx = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$.

Example 8.35. The improper integral $\int_0^{\infty} x\,dx$ diverges to ∞, and the improper integral $\int_{-\infty}^{\infty} (\sin x - 1)\,dx$ diverges to $-\infty$. The improper integral $\int_0^{\infty} \sin x\,dx$ diverges, but not diverges to ∞ or $-\infty$, and the improper integrals $\int_{-\infty}^{\infty} x\,dx$ diverges but not diverges to ∞ or $-\infty$.

Example 8.36. The improper integral $\int_0^{\infty} \frac{\sin x}{x}\,dx$ converges although it is not obvious what its value is. In fact,

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}\,.$$

Theorem 8.37

1. If f is Riemann integrable on the interval $[a, b]$ for all $a < b$, then

$$\int_a^{\infty} f(x)\,dx = \int_a^c f(x)\,dx + \int_c^{\infty} f(x)\,dx \qquad \forall\, a < c\,,$$

provided that the improper integrals on both sides converge or diverge to ∞ (or $-\infty$).

2. If f is Riemann integrable on the interval $[a, b]$ for all $a < b$, then

$$\int_{-\infty}^{b} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{b} f(x)\,dx \qquad \forall\, c < b\,,$$

provided that the improper integrals on both sides converge or diverge to ∞ (or $-\infty$).

3. If f is Riemann integrable on the interval $[a, b]$ for all $a < b$ and $\int_{-\infty}^{\infty} f(x)\,dx$ converges or diverges to ∞ (or $-\infty$), then

$$\int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_b^{\infty} f(x)\,dx \qquad \forall\, a, b \in \mathbb{R}\,.$$