Calculus MA1002-A Quiz 03

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學號：____________________ 姓名：____________________

Problem 1. Let f be a function defined by

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+1.5)} \Big(\frac{x}{2}\Big)^{2k+\frac{1}{2}},$$

where the domain of f is the collection of x such that the infinite series converges, and Γ is the Gamma function. Complete the following.

1. (2%) Is f a power series? Explain your answer.
2. (4%) Find $R > 0$ such that f is defined on $(-R, R)$ but un-defined outside $[-R, R]$. You may need the formula $\Gamma(x+1) = x\Gamma(x)$ for all $x > 0$.
3. (4%) Find the derivative of f on $(-R, R)$.

Solution:

1. No, it is not a power series since it is not an infinite sum of monomials.
2. Since

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\Big|\dfrac{(-1)^{n+1}}{(n+1)!\,\Gamma(n+2.5)}\Big|\Big|\dfrac{x}{2}\Big|^{2(n+1)+\frac{1}{2}}}{\Big|\dfrac{(-1)^n}{n!\,\Gamma(n+1.5)}\Big|\Big|\dfrac{x}{2}\Big|^{2n+\frac{1}{2}}} \\
&= \lim_{n\to\infty} \frac{\Gamma(n+1.5)}{(n+1)\Gamma(n+2.5)}\frac{x^2}{4} = \lim_{n\to\infty} \frac{\Gamma(n+1.5)}{(n+1)(n+1.5)\Gamma(n+1.5)}\frac{x^2}{4} = 0\,,
\end{aligned}$$

   the ratio test implies that the $\sum\limits_{k=0}^{\infty} \dfrac{(-1)^k}{k!\,\Gamma(k+1.5)}\Big(\dfrac{x}{2}\Big)^{2k+\frac{1}{2}}$ converges for all $x \in \mathbb{R}$.

3. Rewrite f as the product of $\sqrt{x}$ and a power series as follows:

$$f(x) = x^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\frac{1}{2}} k!\,\Gamma(k+1.5)} x^{2k} \qquad \forall\, x \in \mathbb{R}\,.$$

   By the product rule and the fact that the derivative of a power series is obtained by term-by-term differentiation, for $x \in \mathbb{R}$ we conclude that

$$\begin{aligned}
f'(x) &= \frac{1}{2}x^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\frac{1}{2}} k!\Gamma(k+1.5)} x^{2k} + x^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{(-1)^k \cdot (2k)}{2^{2k+\frac{1}{2}} k!\Gamma(k+1.5)} x^{2k-1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k+\frac{3}{2}} k!\Gamma(k+1.5)} x^{2k-\frac{1}{2}} + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k-\frac{1}{2}} (k-1)!\Gamma(k+1.5)} x^{2k-\frac{1}{2}}\,.
\end{aligned}$$