Extra Exercise Problem Set 9

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Problem 1. Let $\sum\limits_{k=1}^{\infty} a_k$ be a conditionally convergent series. Show that $\sum\limits_{k=1}^{\infty} \big[1 + \operatorname{sgn}(a_k)\big] a_k$ and $\sum\limits_{k=1}^{\infty} \big[1 - \operatorname{sgn}(a_k)\big] a_k$ both diverge. Here the sign function sgn is defined by

$$\operatorname{sgn}(a) = \begin{cases} 1 & \text{if } a > 0\,, \\ 0 & \text{if } a = 0\,, \\ -1 & \text{if } a < 0\,. \end{cases}$$

Problem 2. A permutation of a non-empty set A is a one-to-one function from A onto A. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$.

1. Suppose that $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence of real numbers. Show that $\big\{a_{\pi(n)}\big\}_{n=1}^{\infty}$ is also convergent; that is, show that if $\{b_n\}_{n=1}^{\infty}$ is a sequence defined by $b_n = a_{\pi(n)}$, then $\{b_n\}_{n=1}^{\infty}$ also converges.

2. Suppose that $\sum\limits_{k=1}^{\infty} a_k$ is absolutely convergent. Show that $\sum\limits_{k=1}^{\infty} a_{\pi(k)}$ is also absolutely convergent, and
$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{\pi(k)}\,.$$

3. Suppose that $\sum\limits_{k=1}^{\infty} a_k$ is conditionally convergent. Show that for each $r \in \mathbb{R}$, there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that
$$\sum_{k=1}^{\infty} a_{\pi(k)} = r\,.$$

Problem 3. In class we have shown that for each $x > 0$,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\,. \tag{$\star$}$$

Show that $(\star)$ in fact holds for all $x \in \mathbb{R}$ (that is, you need to show that $(\star)$ holds for $x \leqslant 0$).

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right) & \text{if } x > 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Find the n-th Maclaurin polynomial for f at 0.