Extra Exercise Problem Set 11

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Problem 1. Prove case 1 of Theorem 9.97 through the following steps.

1. Let $A = \sum_{k=0}^{\infty} a_k r^k$, and define

$$g(x) = f(rx + c) - A = -\sum_{k=1}^{\infty} a_k r^k + \sum_{k=1}^{\infty} a_k r^k x^k = \sum_{k=0}^{\infty} b_k x^k \,,$$

   where $b_k = a_k r^k$ for each $k \in \mathbb{N}$ and $b_0 = -\sum_{k=1}^{\infty} a_k r^k$. Show that the radius of convergence of g is 1 and $\sum_{k=0}^{\infty} b_k = 0$. Moreover, show that f is continuous at $c + r$ if and only if g is continuous at 1.

2. Let $s_n = b_0 + b_1 + \cdots + b_n$ and $S_n(x) = b_0 + b_1 x + \cdots + b_n x^n$. Show that

$$S_n(x) = (1 - x)(s_0 + s_1 x + \cdots + s_{n-1} x^{n-1}) + s_n x^n$$

   and conclude that

$$g(x) = \lim_{n \to \infty} S_n(x) = (1 - x) \sum_{k=0}^{\infty} s_k x^k \,. \qquad (\star)$$

3. Use $(\star)$ to show that g is continuous at 1. Note that you might need to use ε-δ argument.