微積分 MA1001-A 上課筆記（精簡版）

2018.12.25.

Ching-hsiao Arthur Cheng 鄭經斅

Theorem 8.5: Integration by Parts

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du\,.$$

Principles of applying integration by parts: Choose u and v such that $v\,du$ has simpler form than $u\,dv$, and this is usually achieved by

1. finding u such that the derivative of u is a function simpler than u, or
2. finding v such that the derivative of v is more complicate than v.

Using integration by parts, we have shown the following formulas:

$$\int x^r \ln x\,dx = \begin{cases} \dfrac{1}{r+1}x^{r+1}\ln x - \dfrac{1}{(r+1)^2}x^{r+1} + C & \text{if } r \neq -1\,, \\ \dfrac{1}{2}(\ln x)^2 + C & \text{if } r = -1\,. \end{cases}$$

as well as

$$\int e^{ax}\sin(bx)\,dx = \frac{1}{a^2+b^2}\big[ae^{ax}\sin(bx) - be^{ax}\cos(bx)\big] + C\,. \tag{8.2.1}$$

$$\int e^{ax}\cos(bx)\,dx = \frac{1}{a^2+b^2}\big[ae^{ax}\cos(bx) + be^{ax}\sin(bx)\big] + C\,. \tag{8.2.2}$$

Remark 8.12. By the Euler identity

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \forall\,\theta\in\mathbb{R}\,, i=\sqrt{-1}\,, \tag{8.2.3}$$

the desired integrals $\int e^{ax}\sin(bx)\,dx$ and $\int e^{ax}\cos(bx)\,dx$ are the real and imaginary part of the integral $\int e^{ax}e^{ibx}\,dx$. By the fact that $e^{ax}e^{ibx} = e^{(a+ib)x}$ and pretending that $\int e^{cx}\,dx = \frac{1}{c}e^{cx} + C$ for complex number c, we find that

$$\begin{aligned}\int e^{ax}e^{ibx}\,dx &= \frac{1}{a+ib}e^{(a+ib)x} + C = \frac{1}{a+ib}e^{ax}\big[\cos(bx)+i\sin(bx)\big] + C \\ &= \frac{a-ib}{a^2+b^2}e^{ax}\big[\cos(bx)+i\sin(bx)\big] + C \\ &= \frac{e^{ax}}{a^2+b^2}\big[a\cos(bx)+b\sin(bx)+i\big(a\sin(bx)-b\cos(bx)\big)\big] + C\,;\end{aligned}$$

thus we conclude (8.2.1) and (8.2.2).

Example 8.13. Find the indefinite $\int x^n e^{ax}\,dx$, $\int x^n \sin(ax)\,dx$ and $\int x^n \cos(ax)\,dx$, where $a > 0$ is a constant.

Let $u = x^n$ and $v = a^{-1}e^{ax}$ (so that $dv = e^{ax}\,dx$), $v = -a^{-1}\cos(ax)$ (so that $dv = \sin(ax)$) and $v = a^{-1}\sin(ax)$ (so that $dv = \cos(ax)$) in these three cases. Then

$$\int x^n e^{ax}\,dx = \frac{1}{a}x^n e^{ax} - \int \frac{1}{a}e^{ax}\cdot nx^{n-1}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx\,. \tag{8.2.4}$$

Moreover,

$$\begin{aligned}\int x^n \sin(ax)\,dx &= -\frac{1}{a}x^n\cos(ax) + \frac{n}{a}\int x^{n-1}\cos(ax)\,dx\,,\\ \int x^n \cos(ax)\,dx &= \frac{1}{a}x^n\sin(ax) - \frac{n}{a}\int x^{n-1}\sin(ax)\,dx\,.\end{aligned}$$

The two identities above further imply that the following recurrence relations

$$\begin{aligned}\int x^n \sin(ax)\,dx &= -\frac{1}{a}x^n\cos(ax) + \frac{n}{a^2}x^{n-1}\sin(ax) - \frac{n(n-1)}{a^2}\int x^{n-2}\cos(ax)\,dx\,,\\ \int x^n \cos(ax)\,dx &= \frac{1}{a}x^n\sin(ax) + \frac{n}{a^2}x^{n-1}\cos(ax) - \frac{n(n-1)}{a^2}\int x^{n-2}\sin(ax)\,dx\,.\end{aligned}$$

Example 8.14. Using integration by parts, we have

$$\begin{aligned}\int \cos^n x\,dx &= \int \cos^{n-1}x\,d(\sin x) = \sin x\cos^{n-1}x - \int \sin x\,d(\cos^{n-1}x)\\ &= \sin x\cos^{n-1}x + (n-1)\int \sin^2 x\cos^{n-2}x\,dx\\ &= \sin x\cos^{n-1}x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\,dx\\ &= \sin x\cos^{n-1}x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx\,;\end{aligned}$$

thus rearranging terms, we conclude that

$$\int \cos^n x\,dx = \frac{\sin x\cos^{n-1}x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx\,. \tag{8.2.5}$$

Similarly,

$$\int \sin^n x\,dx = -\frac{\cos x\sin^{n-1}x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx\,. \tag{8.2.6}$$

Theorem 8.15: Wallis's Formulas

If n is a non-negative integer, then

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2n+1} x\,dx = \frac{(2^n n!)^2}{(2n+1)!}$$

and

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = \frac{(2n)!}{(2^n n!)^2} \cdot \frac{\pi}{2}\,.$$

Proof. Note that (8.2.5) implies that

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n}\Big|_{x=0}^{x=\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\,dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \cos^{n-2} x\,dx\,.$$

Therefore,

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^{2n+1} x\,dx &= \frac{2n}{2n+1}\int_0^{\frac{\pi}{2}} \cos^{2n-1} x\,dx = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\int_0^{\frac{\pi}{2}} \cos^{2n-3} x\,dx = \cdots \\ &= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\cdots\frac{2}{3}\int_0^{\frac{\pi}{2}} \cos x\,dx = \frac{2}{3}\cdot\frac{4}{5}\cdots\frac{2n}{2n+1} \\ &= \frac{2^2 4^2 \cdots (2n)^2}{(2n+1)!} = \frac{(2^n n!)^2}{(2n+1)!}\end{aligned}$$

and

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx &= \frac{2n-1}{2n}\int_0^{\frac{\pi}{2}} \cos^{2n-2} x\,dx = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\int_0^{\frac{\pi}{2}} \cos^{2n-4} x\,dx = \cdots \\ &= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot\frac{2n-5}{2n-4}\cdots\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^0 x\,dx = \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}\cdot\frac{\pi}{2} \\ &= \frac{(2n)!}{2^2 4^2 \cdots (2n)^2}\cdot\frac{\pi}{2} = \frac{(2n)!}{(2^n n!)^2}\cdot\frac{\pi}{2}\,.\end{aligned}$$

The substitution $x = \frac{\pi}{2} - u$ shows that

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx \qquad \text{for all non-negative integers } n\,,$$

so we conclude the theorem. □

8.3 Trigonometric Integrals

In this section, we are concerned with the integrals

$$\int \sin^m x \cos^n x \, dx \qquad \text{and} \qquad \int \sec^m x \tan^n x \, dx \,,$$

where m, n are non-negative integers.

8.3.1 The integral of $\sin^m x \cos^n x$

• **The case when one of m and n is odd**

Suppose $m = 2k+1$ or $n = 2\ell+1$. Write

$$\int \sin^{2k+1} x \cos^n x \, dx = \int \cos^n x (1-\cos^2 x)^k \sin x \, dx = -\int \cos^n x (1-\cos^x x)^k \, d(\cos x)$$

and

$$\int \sin^m x \cos^{2\ell+1} x \, dx = \int \sin^m x (1-\sin^2 x)^\ell \cos x \, dx = \int \sin^m x (1-\sin^2 x)^\ell \, d(\sin x)$$

so that the integral can be obtained by integrating polynomials.

Example 8.16. Find the indefinite integral $\displaystyle\int \sin^3 x \cos^4 x \, dx$.

Let $u = \cos x$. Then $du = -\sin x \, dx$; thus

$$\begin{aligned}\int \sin^3 x \cos^4 x \, dx &= \int (1-\cos^2 x) \cos^4 x \sin x \, dx = -\int (1-u^2) u^4 \, du \\ &= -\frac{1}{5}u^5 + \frac{1}{7}u^7 + C = -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C \,.\end{aligned}$$

We also write

$$\begin{aligned}\int \sin^3 x \cos^4 x \, dx &= \int (1-\cos^2 x) \cos^4 x \sin x \, dx = -\int (1-\cos^2 x) \cos^4 x \, d(\cos x) \\ &= -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C \,.\end{aligned}$$

• **The case when m and n are both even**

First we talk about how to integrate $\cos^n x$. We have shown the recurrence relation (8.2.5) in previous section, and there are other ways of finding the integral of $\cos^n x$ without using integration by parts. The case when $n = 2\ell + 1$ can be dealt with the previous case, so we focus on the case $n = 2\ell$. Make use of the half angle formula

$$\cos^2 x = \frac{1 + \cos(2x)}{2} ,$$

we can write

$$\int \cos^{2\ell} x\, dx = \int \Big(\frac{1 + \cos(2x)}{2}\Big)^\ell dx = \sum_{i=0}^{\ell} \frac{C^\ell_i}{2^\ell} \int \cos^i(2x)\, dx \overset{(u=2x)}{=} \sum_{i=0}^{\ell} \frac{C^\ell_i}{2^{\ell+1}} \int \cos^i u\, du$$

which is a linear combination of integrals of the form $\displaystyle\int \cos^i u\, du$, while the power i is at most half of n. Keeping on applying the half angle formula for even powers of cosine, eventually integral $\displaystyle\int \cos^i u\, du$ will be reduced to sum of integrals of cosine with odd powers (which can be evaluated by the previous case).

Example 8.17. Find the indefinite integral $\displaystyle\int \cos^6 x\, dx$.

By the half angle formula,

$$\begin{aligned}
\int \cos^6 x\, dx &= \int \Big(\frac{1 + \cos(2x)}{2}\Big)^3 dx = \frac{1}{8} \int \big[1 + 3\cos(2x) + 3\cos^2(2x) + \cos^3(2x)\big]\, dx \\
&= \frac{1}{8} \int \Big[1 + 3\cos(2x) + \frac{3}{2}\big(1 + \cos(4x)\big) + \big(1 - \sin^2(2x)\big)\cos(2x)\Big]\, dx \\
&= \frac{1}{8} \int \Big(\frac{5}{2} + 4\cos(2x) + \frac{3}{2}\cos(4x)\Big)\, dx - \frac{1}{16} \int \sin^2(2x)\, d\big(\sin(2x)\big) \\
&= \frac{1}{8}\Big[\frac{5x}{2} + 2\sin(2x) + \frac{3}{8}\sin(4x)\Big] - \frac{1}{48}\sin^3(2x) + C .
\end{aligned}$$

Now suppose that $m = 2k$ and $n = 2\ell$. Make use of the half angle formulas

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

to write

$$\int \sin^{2k} x \cos^{2\ell} x\, dx = \frac{1}{2^{k+\ell}} \int \big(1 - \cos(2x)\big)^k \big(1 + \cos(2x)\big)^\ell\, dx .$$

Expanding parenthesis, the integral above becomes the linear combination of integrals of the form $\displaystyle\int \cos^i(2x)\, dx$.

Example 8.18. Find the indefinite integral $\displaystyle\int \sin^2 x \cos^4 x\, dx$.

By the half angle formula,

$$\begin{aligned}\int \sin^2 x \cos^4 x\, dx &= \int \frac{1-\cos(2x)}{2}\Big(\frac{1+\cos(2x)}{2}\Big)^2\, dx \\ &= \frac{1}{8}\int \big[1-\cos(2x)\big]\big[1+2\cos(2x)+\cos^2(2x)\big]\, dx \\ &= \frac{1}{8}\int \big[1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big]\, dx \\ &= \frac{1}{8}\int \Big(\frac{1-\cos(4x)}{2}+\sin^2(2x)\cos(2x)\big]\, dx \\ &= \frac{1}{8}\Big[\frac{x}{2}-\frac{\sin(4x)}{8}\Big]+\frac{1}{48}\sin^3(2x)+C\,.\end{aligned}$$

8.3.2 The integral of $\sec^m x \tan^n x$

- **The case when m is even**

Suppose that $m = 0$ and $n \geqslant 2$. Then we obtain the recurrence relation

$$\begin{aligned}\int \tan^n x\, dx &= \int \tan^{n-2} x \tan^2 x\, dx = \int \tan^{n-2}(\sec^2 x - 1)\, dx \\ &= \int \tan^{n-2}\, d(\tan x) - \int \tan^{n-2} x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx\,.\end{aligned}$$

Suppose that $m = 2k$ is even and positive. Using the substitution $u = \tan x$, we have

$$\int \sec^{2k} x \tan^n x\, dx = \int \sec^{2(k-1)} x \tan^n x \sec^2 x\, dx = \int (1+\tan^2 x)^{k-1} \tan^n x\, d(\tan x)$$

which can be obtained by integrating polynomials.