微積分 **MA1001-A** 上課筆記（精簡版）

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Chapter 8

Integration Techniques

8.1 Basic Integration Rules

We recall the following formula:

1. Let f, g be functions and k be a constant. Then

$$\int kf(x)\,dx = k\int f(x)\,dx\,, \qquad \int (f+g)(x)\,dx = \int f(x)\,dx + \int g(x)\,dx\,.$$

2. Let r be a real number. Then

$$\int x^r\,dx = \begin{cases} \dfrac{1}{r+1}x^{r+1} + C & \text{if } r \neq -1\,, \\ \ln x + C & \text{if } r = -1\,. \end{cases}$$

3. If $a > 0$, then $\displaystyle\int a^x\,dx = \frac{1}{\ln a}a^x + C$. In particular, $\displaystyle\int e^x\,dx = e^x + C$.

4. If $a \neq 0$, $\displaystyle\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$, $\displaystyle\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$,
$\displaystyle\int \tan(ax)\,dx = \frac{1}{a}\ln|\sec(ax)| + C$, $\displaystyle\int \cot(ax)\,dx = \frac{1}{a}\ln|\sin(ax)| + C$,
$\displaystyle\int \sec(ax)\,dx = \frac{1}{a}\ln|\sec(ax)+\tan(ax)|+C$, $\displaystyle\int \csc x\,dx = -\frac{1}{a}\ln|\csc(ax)+\cot(ax)|+C$.

5. $\displaystyle\int \sec^2 x\,dx = \tan x + C$, $\displaystyle\int \sec x\tan x\,dx = \sec x + C$.

6. If $a > 0$, then

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C\,, \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arctan\frac{\sqrt{x^2-a^2}}{a} + C\,.$$

Example 8.1. Find the indefinite integrals $\int \frac{4}{x^2+9}\,dx$, $\int \frac{4x}{x^2+9}\,dx$ and $\int \frac{4x^2}{x^2+9}\,dx$.

From the formula above, it is easy to see that

$$\int \frac{4}{x^2+9}\,dx = \frac{4}{3}\arctan\frac{x}{3} + C\,.$$

Noting that $\frac{4x}{x^2+9} = 2\frac{\frac{d}{dx}(x^2+9)}{x^2+9}$, using the formula $\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f'(x)}$, we find that

$$\int \frac{4x}{x^2+9}\,dx = 2\ln(x^2+9) + C\,.$$

Finally, noting that $\frac{4x^2}{x^2+9} = \frac{4(x^2+9)-36}{x^2+9} = 4 - \frac{36}{x^2+9}$, by the formula above we find that

$$\int \frac{4x^2}{x^2+9}\,dx = 4x - 12\arctan\frac{x}{3} + C\,.$$

Example 8.2. Find the indefinite integrals $\int \frac{3}{\sqrt{4-x^2}}\,dx$, $\int \frac{3x}{\sqrt{4-x^2}}\,dx$ and $\int \frac{3x^2}{\sqrt{4-x^2}}\,dx$.

From the formula above,

$$\int \frac{3}{\sqrt{4-x^2}}\,dx = 3\arcsin\frac{x}{2} + C\,.$$

For the second integral, we let $4-x^2 = u$. Then $-2xdx = du$; thus

$$\int \frac{3x}{\sqrt{4-x^2}}\,dx = -\frac{3}{2}\int u^{-\frac{1}{2}}\,du = -\frac{3}{2}\frac{1}{1-\frac{1}{2}}u^{\frac{1}{2}} + C = -3(4-x^2)^{\frac{1}{2}} + C\,.$$

For the third integral, first we observe that

$$\int \frac{3x^2}{\sqrt{4-x^2}}\,dx = \int \frac{3(x^2-4)}{\sqrt{4-x^2}}\,dx + \int \frac{12}{\sqrt{4-x^2}}\,dx = -3\int \sqrt{4-x^2}\,dx + 12\arcsin\frac{x}{2}\,.$$

Let $x = 2\sin u$. Then $dx = 2\cos u\,du$; thus

$$\begin{aligned}\int \sqrt{4-x^2}\,dx &= \int \sqrt{4(1-\sin^2 u)}\cdot 2\cos u\,du = \int 4\cos^2 u\,du = \int \big[2+2\cos(2u)\big]\,du \\ &= 2u + \sin(2u) + C = 2u + 2\sin u\cos u + C \\ &= 2\arcsin\frac{x}{2} + x\sqrt{1-\frac{x^2}{4}} + C = 2\arcsin\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + C\,.\end{aligned}$$

Therefore,

$$\int \frac{3x^2}{\sqrt{4-x^2}}\,dx = 6\arcsin\frac{x}{2} - \frac{3}{2}x\sqrt{4-x^2} + C\,.$$

Remark 8.3. One should add

$$\int \frac{x}{\sqrt{a^2-x^2}}\,dx = -\sqrt{a^2-x^2} + C \qquad \text{and} \qquad \int \frac{x}{\sqrt{a^2+x^2}}\,dx = \sqrt{a^2+x^2} + C$$

into the table of integrations.

Example 8.4. Find the indefinite integral $\displaystyle\int \frac{dx}{1+e^x}$.

Note that

$$\frac{1}{1+e^x} = \frac{1+e^x}{1+e^x} - \frac{e^x}{1+e^x} = 1 - \frac{\frac{d}{dx}(1+e^x)}{1+e^x}\,;$$

thus using the formula $\dfrac{d}{dx}\ln|f(x)| = \dfrac{f'(x)}{f(x)}$, we find that

$$\int \frac{dx}{1+e^x} = x - \ln(1+e^x) + C\,.$$

8.2 Integration by Parts - 分部積分

Suppose that u, v are two differentiable functions of x. Then the product rule implies that

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}\,.$$

Therefore, if $\dfrac{du}{dx}v$ and $u\dfrac{dv}{dx}$ are Riemann integrable (on the interval of interests),

$$\int \frac{du}{dx}v\,dx + \int u\frac{dv}{dx}\,dx = (uv)(x) + C\,.$$

Symbolically, we write $\dfrac{du}{dx}v\,dx$ ad $v\,du$ and $u\dfrac{dv}{dx}\,dx$ as $u\,dv$, the formula above implies that

$$\int udv = uv - \int vdu\,.$$

Theorem 8.5: Integration by Parts

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du\,.$$

Example 8.6. Find the indefinite integral $\int \ln x\,dx$. Recall that we have shown that

$$\int \ln x\,dx = x\ln x - x + C$$

using the Riemann sum. Let $u = \ln x$ and $v = x$ (so that $dv = dx$). Then integration by parts shows that

$$\int \ln x\,dx = x\ln x - \int x\,d(\ln x) = x\ln x - \int x \cdot \frac{1}{x}\,dx = x\ln x - \int dx = x\ln x - x + C\,.$$

Example 8.7. Find the indefinite integral $\int x\cos x\,dx$. Recall that we have shown that

$$\int x\cos x\,dx = x\sin x + \cos x + C$$

using the Riemann sum. Let $u = x$ and $v = \sin x$ (so that $dv = \cos x\,dx$). Then integration by parts shows that

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C\,.$$

Principles of applying integration by parts: Choose u and v such that $v\,du$ has simpler form than $u\,dv$, and this is usually achieved by

1. finding u such that the derivative of u is a function simpler than u, or
2. finding v such that the derivative of v is more complicate than v.

Example 8.8. Find the indefinite integral $\int xe^x\,dx$.

Let $u = x$ and $v = e^x$ (so that $dv = e^x dx$). Then integration by parts shows that

$$\int xe^x\,dx = xe^x - \int e^x\,dx = (x-1)e^x + C\,.$$

Example 8.9. Find the indefinite integral $\int x^r \ln x\,dx$, where r is a real number.

Suppose first that $r \neq -1$. Let $u = \ln x$ and $v = \frac{1}{r+1}x^{r+1}$. Then integration by parts shows that

$$\begin{aligned}\int x^r \ln x\,dx &= \frac{1}{r+1}x^{r+1}\ln x - \int \frac{1}{r+1}x^{r+1} \cdot \frac{1}{x}\,dx = \frac{1}{r+1}x^{r+1}\ln x - \frac{1}{r+1}\int x^r\,dx \\ &= \frac{1}{r+1}x^{r+1}\ln x - \frac{1}{(r+1)^2}x^{r+1} + C\,.\end{aligned}$$

Now if $r = -1$. Let $u = v = \ln x$. Then integration by parts implies that

$$\int x^{-1} \ln x \, dx = (\ln x)^2 - \int \ln x \cdot \frac{1}{x} \, dx = (\ln x)^2 - \int x^{-1} \ln x \, dx$$

which implies that

$$\int x^{-1} \ln x \, dx = \frac{1}{2} (\ln x)^2 + C \,.$$

Therefore,

$$\int x^r \ln x \, dx = \begin{cases} \dfrac{1}{r+1} x^{r+1} \ln x - \dfrac{1}{(r+1)^2} x^{r+1} + C & \text{if } r \neq -1 \,, \\ \dfrac{1}{2} (\ln x)^2 + C & \text{if } r = -1 \,. \end{cases}$$

Example 8.10. Find the indefinite integral $\int x^2 \cos x \, dx$.

Let $u = x^2$ and $v = \sin x$ (so that $dv = \cos x \, dx$). Then integration by parts shows that

$$\int x^2 \cos x \, dx = x^2 \sin x - \int \sin x \cdot 2x \, dx = x^2 \sin x - 2 \int x \sin x \, dx \,.$$

Integrating by parts again, we find that

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C \,;$$

thus we obtain the

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C \,.$$

Example 8.11. Find the indefinite integrals $\int e^{ax} \sin(bx) \, dx$ and $\int e^{ax} \cos(bx) \, dx$, where a, b are non-zero constants.

Let $u = \sin(bx)$ (or $u = \cos(ax)$) and $v = a^{-1} e^{ax}$ (so that $dv = e^{ax} \, dx$). Then

$$\begin{aligned} \int e^{ax} \sin(bx) \, dx &= \frac{1}{a} e^{ax} \sin(bx) - \frac{b}{a} \int e^{ax} \cos(bx) \, dx \,, \\ \int e^{ax} \cos(bx) \, dx &= \frac{1}{a} e^{ax} \cos(bx) + \frac{b}{a} \int e^{ax} \cos(bx) \, dx \,. \end{aligned}$$

The two identities above further imply that

$$\begin{aligned} \int e^{ax} \sin(bx) \, dx &= \frac{1}{a} e^{ax} \sin(bx) - \frac{b}{a} \int e^{ax} \cos(bx) \, dx \\ &= \frac{1}{a} e^{ax} \sin(bx) - \frac{b}{a} \Big[\frac{1}{a} e^{ax} \cos(bx) + \frac{b}{a} \int e^{ax} \sin(bx) \, dx \Big] \\ &= \frac{1}{a} e^{ax} \sin(bx) - \frac{b}{a^2} e^{ax} \cos(bx) - \frac{b^2}{a^2} \int e^{ax} \sin(bx) \, dx \,; \end{aligned}$$

thus

$$\int e^{ax}\sin(bx)\,dx = \frac{1}{a^2+b^2}\big[ae^{ax}\sin(bx) - be^{ax}\cos(bx)\big] + C\,. \tag{8.2.1}$$

Similarly,

$$\int e^{ax}\cos(bx)\,dx = \frac{1}{a^2+b^2}\big[ae^{ax}\cos(bx) + be^{ax}\sin(bx)\big] + C\,. \tag{8.2.2}$$