微積分 MA1001-A 上課筆記（精簡版）

2018.12.11.

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Theorem 5.41: Cauchy Mean Value Theorem

Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}\,.$$

Theorem 5.42: L'Hôspital's Rule

Let f, g be differentiable on (a, b), and $\dfrac{f(x)}{g(x)}$ and $\dfrac{f'(x)}{g'(x)}$ be defined on (a, b). If $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ exists, and one of the following conditions holds:

1. $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = 0$;
2. $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = \infty$,

then $\lim\limits_{x\to a^+} \dfrac{f(x)}{g(x)}$ exists, and

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}\,.$$

Remark 5.43. 1. L'Hôspital Rule can also be applied to the case when $\lim\limits_{x\to b^-}$ replaces $\lim\limits_{x\to a^+}$ in the theorem. Moreover, the one-sided limit can also be replaced by full limit $\lim\limits_{x\to c}$ if $c \in (a, b)$ (by considering L'Hôspital's Rule on (a, c) and (c, b), respectively).

2. L'Hôspital Rule can also be applied to limits as $x \to \infty$ or $x \to -\infty$ (and here a or b has to be changed to $-\infty$ or ∞ as well).

- **Indeterminate form** $\dfrac{0}{0}$

Example 5.44. Compute $\lim\limits_{x\to 0} \dfrac{e^{2x} - 1}{x}$. Last time we conclude from L'Hôspital's Rule that

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = 2 \quad \text{and} \quad \lim_{x\to 0^-} \frac{f(x)}{g(x)} = \lim_{x\to 0^-} \frac{f'(x)}{g'(x)} = 2\,.$$

Theorem 1.26 then shows that $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = 2$ exists.

From the discussion in Example 5.44, using L'Hôspital's Rule in Theorem 5.42 we deduce the following L'Hôspital's Rule for the full limit case.

Theorem 5.42.*

Let $a < c < b$, and f, g be differentiable functions on $(a,b)\backslash\{c\}$. Assume that $g'(x) \neq 0$ for all $x \in (a,b)\backslash\{c\}$. If the limit of $\dfrac{f(x)}{g(x)}$ as x approaches c produces the indeterminate form $\dfrac{0}{0}$ $\Big($or $\dfrac{\infty}{\infty}\Big)$; that is, $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$ $\big($or $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = \infty\big)$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

- **Indeterminate form** $\dfrac{\infty}{\infty}$

Example 5.45. In this example we compute $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}$. Note that $\lim\limits_{x\to\infty}\dfrac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = \lim\limits_{x\to\infty}\dfrac{1}{x} = 0$, so L'Hôspital's Rule implies that

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = 0\,.$$

In fact, the logarithmic function $y = \ln x$ grows slower than any power function; that is,

$$\lim_{x\to\infty}\frac{\ln x}{x^p} = 0 \qquad \forall\, p > 0\,.$$

To see this, note that $\lim\limits_{x\to\infty}\dfrac{\frac{d}{dx}\ln x}{\frac{d}{dx}x^p} = \lim\limits_{x\to\infty}\dfrac{\frac{1}{x}}{px^{p-1}} = \dfrac{1}{p}\lim\limits_{x\to\infty}\dfrac{1}{x^p} = 0$, so L'Hôspital's Rule implies that

$$\lim_{x\to\infty}\frac{\ln x}{x^p} = \lim_{x\to\infty}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x^p} = 0\,.$$

- **Indeterminate form** $0\cdot\infty$

Example 5.46. Compute $\lim\limits_{x\to\infty} e^{-x}\sqrt{x}$. Rewrite $e^{-x}\sqrt{x}$ as $\dfrac{\sqrt{x}}{e^x}$ and note that

$$\lim_{x\to\infty}\frac{\frac{d}{dx}\sqrt{x}}{\frac{d}{dx}e^x} = \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x\to\infty}\frac{1}{2\sqrt{x}e^x} = 0\,.$$

Therefore, L'Hôspital's Rule implies that

$$\lim_{x\to\infty}\frac{\sqrt{x}}{e^x} = \lim_{x\to\infty}\frac{\frac{d}{dx}\sqrt{x}}{\frac{d}{dx}e^x} = 0\,.$$

In fact, the natural exponential function $y = e^x$ grows faster than any power function; that is,

$$\lim_{x\to\infty} \frac{x^p}{e^x} = 0 \qquad \forall\, p > 0\,.$$

The proof is left as an exercise.

- **Indeterminate form** 1^∞

Example 5.47. In this example we compute $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$. Rewrite $(1+x)^{\frac{1}{x}}$ as $e^{\frac{\ln(1+x)}{x}}$. If the limit $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}$ exists, then the continuity of the exponential function implies that

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = \exp\big(\lim_{x\to 0}\frac{\ln(1+x)}{x}\big)\,.$$

Nevertheless, since $\lim\limits_{x\to 0}\ln(1+x) = 0$, $\lim\limits_{x\to 0} x = 0$ and

$$\lim_{x\to 0}\frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x} = \lim_{x\to 0}\frac{1}{1+x} = 1$$

L'Hôspital's Rule implies that

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x} = 1\,;$$

thus $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}} = \exp(1) = e$.

- **Indeterminate form** 0^0

Example 5.48. In this example we compute $\lim\limits_{x\to 0^+}(\sin x)^x$. When $\sin x > 0$, we have

$$(\sin x)^x = e^{x\ln\sin x} = e^{\frac{\ln\sin x}{1/x}}\,.$$

Since

$$\lim_{x\to 0^+}\frac{\frac{d}{dx}\ln\sin x}{\frac{d}{dx}\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = -\lim_{x\to 0^+}\frac{x}{\sin x}x\cos x = 0\,,$$

by L'Hôspital's Rule and the continuity of the natural exponential function we find that

$$\lim_{x\to 0^+}(\sin x)^x = \lim_{x\to 0^+} e^{\frac{\ln\sin x}{1/x}} = e^0 = 1\,.$$

- **Indeterminate form $\infty - \infty$**

Example 5.49. Compute $\lim_{x\to 1+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$.

Rewrite $\frac{1}{\ln x} - \frac{1}{x-1} = \frac{x-1-\ln x}{(x-1)\ln x}$ and note that the right-hand side produces indeterminate form $\frac{0}{0}$ as x approaches from the right. Also note that

$$\frac{\frac{d}{dx}(x-1-\ln x)}{\frac{d}{dx}(x-1)\ln x} = \frac{1-\frac{1}{x}}{\ln x + \frac{x-1}{x}} = \frac{x-1}{x\ln x + x - 1}$$

which, as x approaches 1 from the right, again produces indeterminate form $\frac{0}{0}$. In order to find the limit of the right-hand side we compute

$$\lim_{x\to 1^+}\frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(x\ln x + x - 1)} = \lim_{x\to 1^+}\frac{1}{\ln x + 1 + 1} = \frac{1}{2};$$

thus L'Hôspital's Rule implies that

$$\lim_{x\to 1^+}\frac{x-1}{x\ln x + x - 1} = \lim_{x\to 1^+}\frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(x\ln x + x - 1)} = \frac{1}{2}.$$

This in turm shows that

$$\lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x} = \lim_{x\to 1^+}\frac{\frac{d}{dx}(x-1-\ln x)}{\frac{d}{dx}(x-1)\ln x} = \lim_{x\to 1^+}\frac{x-1}{x\ln x + x - 1} = \frac{1}{2}.$$

5.7 The Inverse Trigonometric Functions: Differentiation

Definition 5.50

The arcsin, arccos, and arctan functions are the inverse functions of the function $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$, $g : [0, \pi] \to \mathbb{R}$, and $h : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, respectively, where $f(x) = \sin x$, $g(x) = \cos x$ and $h(x) = \tan x$. In other words,

1. $y = \arcsin x$ if and only if $\sin y = x$, where $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, $-1 \leqslant x \leqslant 1$.
2. $y = \arccos x$ if and only if $\cos y = x$, where $0 \leqslant y \leqslant \pi$, $-1 \leqslant x \leqslant 1$.
3. $y = \arctan x$ if and only if $\tan y = x$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$, $-\infty < x < \infty$.

Remark 5.51. Since arcsin, arccos and arctan look like the inverse function of sin, cos and tan, respectively, often times we also write arcsin as $\sin^{-1}$, arccos as $\cos^{-1}$, and arctan as $\tan^{-1}$.

Example 5.52. $\arcsin\frac{1}{2} = \frac{\pi}{6}$, $\arccos\left(\frac{-\sqrt{2}}{2}\right) = \frac{3\pi}{4}$, and $\arctan 1 = \frac{\pi}{4}$.

Example 5.53. Suppose that $y = \arcsin x$. Then $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which implies that $\cos y \geqslant 0$. Therefore, by the fact that $\sin^2 y + \cos^2 y = 1$, we have

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \qquad \text{if} \quad y = \arcsin x\,.$$

Similarly, if $y = \arccos x$, then $y \in (0, \pi)$ which implies that $\sin y \geqslant 0$. Therefore,

$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2} \qquad \text{if} \quad y = \arccos x\,.$$