微積分 MA1001-A 上課筆記（精簡版）

2018.11.29.

Ching-hsiao Arthur Cheng 鄭經斅

Definition 5.8

The function $\ln : (0, \infty) \to \mathbb{R}$ is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt \qquad \forall\, x > 0\,.$$

Theorem 5.10

$\dfrac{d}{dx} \ln x = \dfrac{1}{x}$ for all $x > 0$.

Corollary 5.11

The function $\ln : (0, \infty) \to \mathbb{R}$ is strictly increasing on $(0, \infty)$, and the graph of $y = \ln x$ is concave downward on $(0, \infty)$.

• The range of $y = \ln x$ is $\mathbb{R}$; thus combining with the corollary above, we have

$$\ln : (0, \infty) \to \mathbb{R} \text{ is one-to-one and onto.}$$

Moreover, there exists a unique $e \in (2, 3)$ such that $\ln e = 1$.

• **Logarithmic Laws**

The function $y = \ln x$ is in fact the logarithmic function to the base e; that is, $\ln = \log_e$, so we have the following

Theorem 5.14: Logarithmic properties of $y = \ln x$

Let a, b be positive numbers and r is rational. Then

1. $\ln 1 = 0$;
2. $\ln(ab) = \ln a + \ln b$;
3. $\ln(a^r) = r \ln a$;
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.

Theorem 5.17

If f is a differentiable function on an interval I, then $\ln |f|$ is differentiable at those point $x \in I$ satisfying $f(x) \neq 0$. Moreover,

$$\frac{d}{dx} \ln \big|f(x)\big| = \frac{f'(x)}{f(x)} \qquad \text{for all } x \in I \text{ with } f(x) \neq 0\,.$$

5.3 Integrations Related to $y = \ln x$

Theorem 5.17 implies the following

Theorem 5.20

1. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$; 2. $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln\big|f(x)\big| + C$.

Example 5.21. Compute $\displaystyle\int \frac{x}{x^2+1}\,dx$. From observation, the numerator is a half of the derivative of the denominator, so

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln|x^2+1| + C = \frac{1}{2}\ln(x^2+1) + C\,.$$

Example 5.22. Compute $\displaystyle\int \frac{1}{x\ln x}\,dx$. Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$; thus

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln x| + C\,.$$

Theorem 5.23

1. $\displaystyle\int \sin x\,dx = -\cos x + C$; 2. $\displaystyle\int \cos x\,dx = \sin x + C$;

3. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C$;

4. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$.

Proof. We only prove 4. Let $t = \tan\frac{x}{2}$. Then $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$ and $dx = \dfrac{2dt}{1+t^2}$; thus

$$\begin{aligned}\int \sec x\,dx &= \int \frac{1+t^2}{1-t^2}\frac{2}{1+t^2}\,dt = \int \frac{2}{1-t^2}\,dt = \int \frac{-2}{(t-1)(t+1)}\,dt \\ &= \int \Big[\frac{1}{t+1} - \frac{1}{t-1}\Big]\,dt = \ln|t+1| - \ln|t-1| + C = \ln\Big|\frac{t+1}{t-1}\Big| + C\,.\end{aligned}$$

The conclusion then follows from the identity

$$\begin{aligned}\frac{t+1}{t-1} &= \frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\sin\frac{x}{2}-\cos\frac{x}{2}} = \frac{\big(\sin\frac{x}{2}+\cos\frac{x}{2}\big)^2}{\sin^2\frac{x}{2}-\cos^2\frac{x}{2}} = \frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{-\cos x} \\ &= -\frac{1+\sin x}{\cos x} = -(\sec x + \tan x)\,.\end{aligned}$$

□

Finally we compute $\int_1^a \ln x\,dx$ for $a > 0$. Suppose first that $a > 1$. Following the idea of Example 4.5, we let $r = a^{\frac{1}{n}}$ and $x_i = r^i$, as well as a partition $\mathcal{P} = \{1 = x_0 < x_1 < \cdots < x_n = a\}$ of $[1, a]$. Then the Riemann sum of f for the partition $\mathcal{P}$ given by the right end-point rule, which happens to be the upper sum of f for the partition $\mathcal{P}$, is

$$S(\mathcal{P}) = \sum_{i=1}^{n} \ln(x_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \ln(r^i)(r^i - r^{i-1}) = (r-1)\ln r \sum_{i=1}^{n} i r^{i-1}\,.$$

Note that $ir^{i-1} = \frac{d}{dr} r^i$; thus

$$\begin{aligned}\sum_{i=1}^{n} i r^{i-1} &= \sum_{i=1}^{n} \frac{d}{dr} r^i = \frac{d}{dr} \sum_{i=1}^{n} r^i = \frac{d}{dr} \frac{r^{n+1} - r}{r-1} = \frac{\big[(n+1)r^n - 1\big](r-1) - r^{n+1} + r}{(r-1)^2} \\ &= \frac{nr^{n+1} - (n+1)r^n + 1}{(r-1)^2} = \frac{nar - (n+1)a + 1}{(r-1)^2}\,.\end{aligned}$$

By the fact that $n = \frac{\ln a}{\ln r}$,

$$S(\mathcal{P}) = \frac{ra\ln a - a\ln a - a\ln r + \ln r}{r-1}\,.$$

Since $\|\mathcal{P}\| \to 0$ is equivalent to that $r \to 1$,

$$\begin{aligned}\lim_{\|\mathcal{P}\|\to 0} S(\mathcal{P}) &= \lim_{r\to 1} \frac{ra\ln a - a\ln a - a\ln r + \ln r}{r-1} = \frac{d}{dr}\Big|_{r=1} \big(ra\ln a - a\ln a - a\ln r + \ln r\big) \\ &= a\ln a - a + 1\,.\end{aligned}$$

If $0 < a < 1$, by Remark 4.16 it suffices to show that $a^{\frac{1}{n}} \to 1$ as n approaches infinity. Nevertheless, $a^{\frac{1}{n}} = 1/(1/a)^{\frac{1}{n}}$ and the denominator approaches 1 as n approaches infinity; thus $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ even if $0 < a < 1$.

Theorem 5.24

1. $\int_1^a \ln x\,dx = a\ln a - a + 1$ for all $a > 0$;
2. $\int \ln x\,dx = x\ln x - x + C$.

5.4 Exponential Functions

In the previous section we have shown that the natural logarithmic function $\ln : (0, \infty) \to \mathbb{R}$ is one-to-one and onto. Therefore, for each $a \in \mathbb{R}$ there exists a unique $b \in (0, \infty)$ satisfying $a = \ln b$. The map $a \mapsto b$ is called the natural exponential function. To be more precise, we have the following

Definition 5.25

The natural exponential function $\exp : \mathbb{R} \to (0, \infty)$ is a function defined by

$$\exp(x) = y \qquad \text{if and only if} \qquad x = \ln y\,.$$

By the definition of the natural exponential function, we have

$$\exp(\ln x) = x \quad \forall\, x \in (0, \infty) \qquad \text{and} \qquad \ln(\exp(x)) = x \quad \forall\, x \in \mathbb{R}\,. \tag{5.4.1}$$

Therefore, exp and ln are inverse functions to each other; thus $\exp : \mathbb{R} \to (0, \infty)$ is one-to-one, onto, and strictly increasing. Note that by the definition, $\exp(0) = 1$.

Let $a > 0$ be a real number. If $r \in \mathbb{Q}$, a^r is a well-defined positive number and the logarithmic laws implies that

$$\ln a^r = r \ln a\,.$$

By the definition of the natural exponential function, $a^r = \exp(r \ln a)$ for all $r \in \mathbb{Q}$. Since $\exp : \mathbb{R} \to (0, \infty)$ is continuous, for a real number x, we shall defined a^x as $\exp(x \ln a)$ and this induces the following

Definition 5.26

Let $a > 0$ be a real number. For each $x \in \mathbb{R}$, the exponential function to the base a, denote by $y = a^x$, is defined by $a^x \equiv \exp(x \ln a)$. In other words,

$$a^x = \exp(x \ln a) \qquad \forall\, x \in \mathbb{R}\,.$$

Remark 5.27. For each $x \in \mathbb{R}$, the number 1^x is 1 since $1^x = \exp(x \ln 1) = \exp(0) = 1$.

Remark 5.28. The function $y = e^x$ is identical to the function $y = \exp(x)$ since

$$e^x = \exp(x \ln e) = \exp(x) \qquad \forall\, x \in \mathbb{R}\,.$$

Therefore, we often write $\exp(x)$ as e^x as well (even though e^x, when x is a irrational number, has to be defined through the natural exponential function).

Remark 5.29. By the definition of the natural exponential function,

$$\ln(a^x) = \ln(\exp(x \ln a)) = x \ln a \qquad \forall\, a > 0 \text{ and } x \in \mathbb{R}\,. \tag{5.4.2}$$

5.4.1 Properties of Exponential Functions

- **The law of exponentials**

(a) If $a > 0$, then $a^{x+y} = a^x a^y$ for all $x, y \in \mathbb{R}$: First we show the case when $a = e$. Let $\exp(x) = c$ and $\exp(y) = d$ or equivalently, $x = \ln c$ and $y = \ln d$. Then

$$e^{x+y} = \exp(x + y) = \exp(\ln c + \ln d) = \exp(\ln(cd)) = cd = e^x e^y\,.$$

For general $a > 0$, by the definition of exponential functions,

$$a^{x+y} = e^{(x+y)\ln a} = e^{x \ln a + y \ln a} = e^{x \ln a} e^{y \ln a} = a^x a^y \qquad \forall\, x, y \in \mathbb{R}\,.$$