微積分 MA1001-A 上課筆記（精簡版）

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Definition 5.8

The function $\ln : (0,\infty) \to \mathbb{R}$ is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad \forall\, x > 0\,.$$

Theorem 5.10

$\dfrac{d}{dx}\ln x = \dfrac{1}{x}$ for all $x > 0$.

Corollary 5.11

The function $\ln : (0,\infty) \to \mathbb{R}$ is strictly increasing on $(0,\infty)$, and the graph of $y = \ln x$ is concave downward on $(0,\infty)$.

We also show that

$$x - \frac{x^2}{2} \leqslant \ln(1+x) \leqslant x \qquad \forall\, x > 0\,. \tag{5.2.1}$$

- **The range**

Next we show that $\lim\limits_{x\to\infty} \ln x = \infty$ and $\lim\limits_{x\to-\infty} \ln x = -\infty$. To see this, we note that

$$\begin{aligned}\ln(2^n) &= \int_1^{2^n} \frac{1}{t}\,dt = \int_1^2 \frac{1}{t}\,dt + \int_2^4 \frac{1}{t}\,dt + \int_4^8 \frac{1}{t}\,dt + \cdots + \int_{2^{n-1}}^{2^n} \frac{1}{t}\,dt\\ &= \sum_{i=1}^n \int_{2^{i-1}}^{2^i} \frac{1}{t}\,dt \geqslant \sum_{i=1}^n \int_{2^{i-1}}^{2^i} \frac{1}{2^i}\,dt = \sum_{i=1}^n \frac{2^i - 2^{i-1}}{2^i} = \sum_{i=1}^n \frac{1}{2} = \frac{n}{2}\end{aligned}$$

and

$$\begin{aligned}\ln(2^{-n}) &= \int_1^{2^{-n}} \frac{1}{t}\,dt = -\int_{2^{-n}}^1 \frac{1}{t}\,dt = -\Big[\int_{2^{-n}}^{2^{-n+1}} \frac{1}{t}\,dt + \int_{2^{-n+1}}^{2^{-n+2}} \frac{1}{t}\,dt + \cdots + \int_{\frac{1}{2}}^1 \frac{1}{t}\,dt\Big]\\ &= -\sum_{i=1}^n \int_{2^{-i}}^{2^{1-i}} \frac{1}{t}\,dt \leqslant -\sum_{i=1}^n \int_{2^{-i}}^{2^{1-i}} \frac{1}{2^{1-i}}\,dt = -\sum_{i=1}^n \frac{2^{1-i} - 2^{-i}}{2^{1-i}} = -\sum_{i=1}^n \frac{1}{2} = -\frac{n}{2}\,;\end{aligned}$$

thus we have $\lim\limits_{x\to\infty} \ln x = \infty$ and $\lim\limits_{x\to-\infty} \ln x = -\infty$. By the continuity of ln and the Intermediate Value Theorem, for each $b \in \mathbb{R}$ there exists one $a \in (0,\mathbb{R})$ such that $b = \ln a$. By the strict monotonicity $\ln : (0,\infty) \to \mathbb{R}$ is one-to-one and onto.

Remark 5.13. In particular, there exists one unique number e such that $\ln e = 1$. We note that

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt = \int_1^{1.5} \frac{1}{t}\,dt + \int_{1.5}^2 \frac{1}{t}\,dt \leqslant \frac{0.5}{1} + \frac{0.5}{1.5} = \frac{5}{6} < 1$$

and

$$\begin{aligned}\ln 3 &= \int_1^3 \frac{1}{t}\,dt = \Big(\int_1^{1.25} + \int_{1.25}^{1.5} + \int_{1.5}^{1.75} + \int_{1.75}^{2} + \int_2^{2.5} + \int_{2.5}^{3}\Big)\frac{1}{t}\,dt \\ &\geqslant \frac{0.25}{1.25} + \frac{0.25}{1.5} + \frac{0.25}{1.75} + \frac{0.25}{2} + \frac{0.5}{2.5} + \frac{0.5}{3} \\ &= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{5} + \frac{1}{6} = \frac{841}{840} > 1\,.\end{aligned}$$

Therefore, $2 < e < 3$.

- **Logarithmic Laws**

The most important property of the function $y = \ln x$ is the relation among $\ln a$, $\ln b$ and $\ln(ab)$. By the property of integration,

$$\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt = \ln a + \int_a^{ab} \frac{1}{t}\,dt\,.$$

By the substitution $t = au$, $dt = adu$; thus

$$\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{au} adu = \int_1^b \frac{1}{u}\,du = \ln b\,.$$

Therefore, we obtain the identity:

$$\ln(ab) = \ln a + \ln b \qquad \forall\, a, b > 0\,. \tag{5.2.2}$$

Having established (5.2.2), we can show that the function ln is a logarithmic function for the following reason. First, we observe that for all $a > 0$ and $n \in \mathbb{N}$,

$$\ln(a^n) = \ln(a^{n-1}a) = \ln(a^{n-1}) + \ln a = \ln(a^{n-2}a) + \ln a = \ln(a^{n-2}) + 2\ln a = \cdots = n\ln a\,.$$

Moreover, by the definition of ln, $0 = \ln(1) = \ln(a^0) = 0\ln a$; thus

$$\ln(a^n) = n\ln a \qquad \forall\, a > 0\,, n \in \mathbb{N} \cup \{0\}\,.$$

Next, by the law of exponents, for $a > 0$ and $n \in \mathbb{N}$ we have

$$0 = \ln(a^0) = \ln(a^n \cdot a^{-n}) = \ln(a^n) + \ln(a^{-n}) = n\ln a + \ln(a^{-n})\,.$$

Therefore, for all $n \in \mathbb{N}$, we also have $\ln(a^{-n}) = -n \ln a$; hence

$$\ln(a^n) = n \ln a \qquad \forall\, a > 0\,, n \in \mathbb{Z}\,.$$

The identity above also implies that if $k, n \in \mathbb{Z}$ and $n \neq 0$,

$$n \ln(a^{\frac{k}{n}}) = \ln((a^{\frac{k}{n}})^n) = \ln(a^k) = k \ln a\,,$$

and this shows that

$$\ln(a^{\frac{k}{n}}) = \frac{k}{n} \ln a \qquad \forall\, a > 0\,, n, k \in \mathbb{Z}\,, n \neq 0\,.$$

As a consequence,

$$\ln(a^r) = r \ln a \qquad \forall\, a > 0\,, r \in \mathbb{Q}\,.$$

Finally, we find that $\ln(e^r) = r \ln e = r$, so $\ln x$ is indeed the logarithm of x to base e. In other words, we obtain that

$$\log_e x = \ln x = \int_1^x \frac{1}{t}\, dt \qquad \forall\, x > 0\,. \tag{5.2.3}$$

Theorem 5.14: Logarithmic properties of $y = \ln x$

Let a, b be positive numbers and r be a rational number. Then

1. $\ln 1 = 0$;
2. $\ln(ab) = \ln a + \ln b$;
3. $\ln(a^r) = r \ln a$;
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.

Remark 5.15. Since the function $y = \ln x$ has the logarithmic property, it is called the **natural logarithmic** function.

Example 5.16. $\ln \dfrac{(x^2+3)^2}{x\sqrt[3]{x^2+1}} = 2\ln(x^2+3) - \ln x - \dfrac{1}{3}\ln(x^2+1)$ if $x > 0$.

Theorem 5.17

If f is a differentiable function on an interval I, then $\ln|f|$ is differentiable at those point $x \in I$ satisfying $f(x) \neq 0$. Moreover,

$$\frac{d}{dx} \ln\big|f(x)\big| = \frac{f'(x)}{f(x)} \qquad \text{for all } x \in I \text{ with } f(x) \neq 0\,.$$

Proof. Note that the function $y = |x|$ is differentiable at non-zero points, and

$$\frac{d}{dx}|x| = \frac{d}{dx}(x^2)^{\frac{1}{2}} = \frac{1}{2}(x^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{|x|} \qquad \forall\, x \neq 0\,.$$

If $f(c) \neq 0$, by the fact that the natural logarithmic function ln is differentiable at $|f(c)|$, the absolute function $|\cdot|$ is differentiable at $f(c)$ and f is differentiable at c, the chain rule implies that $y = \ln\big|f(x)\big|$ is differentiable at c and

$$\frac{d}{dx}\Big|_{x=c} \ln\big|f(x)\big| = \frac{1}{\big|f(c)\big|}\frac{f(c)}{\big|f(c)\big|}f'(c) = \frac{f'(c)}{f(c)}\,. \qquad \square$$

Example 5.18. $\frac{d}{dx}\ln|\cos x| = \frac{-\sin x}{\cos x} = -\tan x$ for all x with $\cos x \neq 0$.

Example 5.19. Compute the derivative of $f(x) = \frac{(x^2+3)^2}{x\sqrt[3]{x^2+1}}$ for $x > 0$.

Let $h(x) = \ln f(x)$. Then

$$\begin{aligned}\frac{f'(x)}{f(x)} = h'(x) = \frac{d}{dx}\Big[2\ln(x^2+3) - \ln x - \frac{1}{3}\ln(x^2+1)\Big] \\ = 2\frac{d}{dx}\ln(x^2+3) - \frac{d}{dx}\ln x - \frac{1}{3}\frac{d}{dx}\ln(x^2+1) \\ = \frac{4x}{x^2+3} - \frac{1}{x} - \frac{2x}{3(x^2+1)}\,;\end{aligned}$$

thus

$$f'(x) = \frac{(x^2+3)^2}{x\sqrt[3]{x^2+1}}\Big[\frac{4x}{x^2+3} - \frac{1}{x} - \frac{2x}{3(x^2+1)}\Big]\,.$$