# 微積分 MA1001-A 上課筆記（精簡版）

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**Definition 4.22**

A function $F$ is an anti-derivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

**Theorem 4.23**

If $F$ is an anti-derivative of $f$ on an interval $I$, then $G$ is an anti-derivative of $f$ on the interval $I$ if and only if $G$ is of the form $G(x) = F(x) + C$ for all $x$ in $I$, where $C$ is a constant.（導函數相同的函數相差一常數）

**Theorem 4.24: Mean Value Theorem for Integrals - 積分均值定理**

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)\,.$$

**Theorem 4.25: Fundamental Theorem of Calculus - 微積分基本定理**

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and $F$ be an anti-derivative of $f$ on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Moreover, if $G(x) = \displaystyle\int_a^x f(t)\,dt$ for $x \in [a, b]$, then $G$ is an anti-derivative of $f$.

**Example 4.32.** Find $\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}} \sin^{100} t\,dt$ for $x > 0$.

Let $F(x) = \displaystyle\int_0^x \sin^{100} t\,dt$. Then by the chain rule,

$$\frac{d}{dx}F(\sqrt{x}) = F'(\sqrt{x})\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}F'(\sqrt{x})\,.$$

By the Fundamental Theorem of Calculus, $F'(x) = \sin^{100} x$; thus

$$\frac{d}{dx}\int_0^{\sqrt{x}} \sin^{100} t\,dt = \frac{d}{dx}F(\sqrt{x}) = \frac{\sin^{100}\sqrt{x}}{2\sqrt{x}}\,.$$

**Theorem 4.28**

Let $f : [a, b] \to \mathbb{R}$ be continuous and $f$ is differentiable on $(a, b)$. If $f'$ is Riemann integrable on $[a, b]$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a)\,.$$

**Theorem 4.25 and 4.28 can be combined as follows:**

**Theorem 4.31**

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function and $F$ be an anti-derivative of $f$ on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Moreover, if in addition $f$ is continuous on $[a, b]$, then $G(x) = \int_a^x f(t)\,dt$ is differentiable on $[a, b]$ and

$$G'(x) = f(x) \qquad \text{for all } x \in [a, b]\,.$$

**Definition 4.30**

An anti-derivative of $f$, if exists, is denoted by $\int f(x)\,dx$, and sometimes is also called an indefinite integral of $f$.

- Basic Rules of Integration:

| Differentiation Formula | Anti-derivative Formula |
|---|---|
| $\frac{d}{dx}C = 0$ | $\int 0\,dx = C$ |
| $\frac{d}{dx}x^r = rx^{r-1}$ | $\int x^q\,dx = \frac{x^{q+1}}{q+1} + C \quad \text{if } q \neq -1$ |
| $\frac{d}{dx}\sin x = \cos x$ | $\int \cos x\,dx = \sin x + C$ |
| $\frac{d}{dx}\cos x = -\sin x$ | $\int \sin x\,dx = -\cos x + C$ |
| $\frac{d}{dx}\tan x = \sec^2 x$ | $\int \sec^2 x\,dx = \tan x + C$ |
| $\frac{d}{dx}\sec x = \sec x \tan x$ | $\int \sec x \tan x\,dx = \sec x + C$ |
| $\frac{d}{dx}\big[kf(x) + g(x)\big] = kf'(x) + g'(x)$ | $\int \big[kf'(x) + g'(x)\big]\,dx = kf(x) + g(x) + C$ |

## 4.4 Integration by Substitution - 變數變換

Suppose that $g : [a, b] \to \mathbb{R}$ is one-to-one and differentiable, and $f : \text{range}(g) \to \mathbb{R}$ is differentiable. Then the chain rule implies that $f \circ g$ is an anti-derivative of $(f' \circ g)g'$; thus provided that

1. $(f \circ g)'$ is Riemann integrable on $[a, b]$,

2. $f'$ is Riemann integrable on the range of $g$,

then Theorem 4.28 implies that

$$\begin{aligned}\int_a^b f'\big(g(x)\big)g'(x)\,dx &= \int_a^b (f \circ g)'(x)\,dx = (f \circ g)(b) - (f \circ g)(a) \\ &= f\big(g(b)\big) - f\big(g(a)\big) = \int_{g(a)}^{g(b)} f'(u)\,du\,. \end{aligned} \tag{4.4.1}$$

Replacing $f'$ by $f$ in the identity above shows the following

**Theorem 4.33**

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$, and $f$ is continuous on the range of $g$, then

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du\,.$$

The anti-derivative version of Theorem 4.33 is stated as follows.

**Theorem 4.34**

Let $g$ be a function with range $I$ and $f$ be a continuous function on $I$. If $g$ is differentiable on its domain and $F$ is an anti-derivative of $f$ on $I$, then

$$\int f\big(g(x)\big)g'(x)\,dx = F\big(g(x)\big) + C$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C\,.$$

**Example 4.35.** Find $\displaystyle\int (x^2+1)^2(2x)\,dx$.

Let $u = x^2 + 1$. Then $du = 2xdx$; thus

$$\int (x^2+1)^2(2x)\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(x^2+1)^3 + C\,.$$

**Example 4.36.** Find $\displaystyle\int \cos(5x)\,dx$.

Let $u = 5x$. Then $du = 5dx$; thus

$$\int \cos(5x)\,dx = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin(5x) + C\,.$$

**Example 4.37.** Find $\displaystyle\int \sec^2 x(\tan x + 3)\,dx$.

Let $u = \tan x$. Then $du = \sec^2 x dx$; thus

$$\int \sec^2 x(\tan x + 3)\,dx = \int (u+3)\,du = \frac{1}{2}u^2 + 3u + C = \frac{1}{2}\tan^2 x + 3\tan x + C\,.$$

On the other hand, let $v = \tan x + 3$. Then $dv = \sec^2 x\,dx$; thus

$$\begin{aligned}\int \sec^2 x(\tan x + 3)\,dx &= \int v\,dv = \frac{1}{2}v^2 + C = \frac{1}{2}(\tan x + 3)^2 + C\\ &= \frac{1}{2}\tan^2 x + 3\tan x + \frac{9}{2} + C\,.\end{aligned}$$

We note that even though the right-hand side of the two indefinite integrals look different, they are in fact the same since $C$ could be any constant, and $\frac{9}{2} + C$ is also any constant.

# Chapter 5. Logarithmic, Exponential, and other Transcendental Functions

## 5.1 Inverse Functions（課本 §5.3）

**Definition 5.1**

A function $g$ is the inverse function of the function $f$ if

$$f(g(x)) = x \quad \text{for all } x \text{ in the domain of } g \tag{5.1.1}$$

and

$$g(f(x)) = x \quad \text{for all } x \text{ in the domain of } f. \tag{5.1.2}$$

The inverse function of $f$ is usually denoted by $f^{-1}$.

**Some important observations about inverse functions**:

1. If $g$ is the inverse function of $f$, then $f$ is the inverse function of $g$.

2. Note that (5.1.1) implies that

   (a) the domain of $g$ is contained in the range of $f$,

   (b) the domain of $f$ contains the range of $g$,

   (c) $g$ is one-to-one since if $g(x_1) = g(x_2)$, then $x_1 = f(g(x_1)) = f(g(x_2)) = x_2$

   and (5.1.2) implies that

   (a) the domain of $f$ is contained in the range of $g$,

   (b) the domain of $g$ contains the range of $f$,

   (c) $f$ is one-to-one since if $f(x_1) = f(x_2)$, then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$.

   According to the statements above, the domain of $f^{-1}$ is the range of $f$, and the range of $f^{-1}$ is the domain of $f$.