微積分 MA1001-A 上課筆記（精簡版）

2018.11.13.

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Theorem 1.45: Intermediate Value Theorem - 中間值定理

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

Theorem 3.8: Mean Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable on (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Definition 4.7: Riemann Integrals - 黎曼積分

Let $f : [a, b] \to \mathbb{R}$ be a function. f is said to be Riemann integrable on $[a, b]$ if there exists a real number A such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums for the partition $\mathcal{P}$ belongs to the interval $(A - \varepsilon, A + \varepsilon)$. Such a number A (is unique and) is called the Riemann integral of f on $[a, b]$ and is denoted by $\int_{[a,b]} f(x)\,dx$.

4.3 The Fundamental Theorem of Calculus

In this section, we develop a theory which shows a systematic way of finding integrals if the integrand is a continuous function.

Definition 4.22

A function F is an anti-derivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem 4.23

If F is an anti-derivative of f on an interval I, then G is an anti-derivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$ for all x in I, where C is a constant.（導函數相同的函數相差一常數）

Proof. It suffices to show the "$\Rightarrow$" (only if) direction. Suppose that $F' = G' = f$ on I. Then the function $h = F - G$ satisfies $h'(x) = 0$ for all $x \in I$. By the mean value theorem, for any $a, b \in I$ with $a \neq b$, there exists c in between a and b such that

$$h(b) - h(a) = h'(c)(b-a)\,.$$

Since $h'(x) = 0$ for all $x \in I$, $h(a) = h(b)$ for all $a, b \in I$; thus h is a constant function. □

Theorem 4.24: Mean Value Theorem for Integrals - 積分均值定理

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)\,.$$

Proof. By the Extreme Value Theorem, f has a maximum and a minimum on $[a, b]$. Let $M = f(x_1)$ and $m = f(x_2)$, where $x_1, x_2 \in [a, b]$, denote the maximum and minimum of f on $[a, b]$, respectively. Then $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$; thus Corollary 4.20 implies that

$$m(b-a) = \int_a^b m\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b M\,dx = M(b-a)\,.$$

Therefore, the number $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx \in [m, M]$. By the Intermidiate Value Theorem, there exists c in between x_1 and x_2 such that $f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$. □

Theorem 4.25: Fundamental Theorem of Calculus - 微積分基本定理

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and F be an anti-derivative of f on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Moreover, if $G(x) = \displaystyle\int_a^x f(t)\,dt$ for $x \in [a, b]$, then G is an anti-derivative of f.

We note that for $x \in [a, b]$, f is continuous on $[a, x]$; thus f is Riemann integrable on $[a, x]$ which shows that $G(x) = \displaystyle\int_a^x f(t)\,dt$ is well-defined.

Proof of the Fundamental Theorem of Calculus. Note that for $h \neq 0$ such that $x + h \in [a, b]$, we have

$$\frac{G(x+h) - G(x)}{h} = \frac{1}{h}\Big[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\Big] = \frac{1}{h}\int_x^{x+h} f(t)\,dt\,.$$

By the Mean Value Theorem for Integrals, there exists $c = c(h)$ in between x and $x+h$ such that $\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c)$. Since f is continuous on $[a,b]$, $\lim\limits_{h\to 0} f(c) = \lim\limits_{c\to x} f(c) = f(x)$; thus

$$\lim_{h\to 0}\frac{G(x+h)-G(x)}{h} = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c) = f(x)$$

which shows that G is an anti-derivative of f on $[a,b]$.

By Theorem 4.23, $G(x) = F(x) + C$ for all $x \in [a,b]$. By the fact that $G(a) = 0$, $C = -F(a)$; thus

$$\int_a^b f(x)\,dx = G(b) = F(b) - F(a)$$

which concludes the theorem. □

Example 4.26. Since an anti-derivative of the function $y = x^q$, where $q \neq -1$ is a rational number, is $y = \frac{x^{q+1}}{q+1}$, we find that

$$\int_a^b x^q\,dx = \frac{x^{q+1}}{q+1}\Big|_{x=b} - \frac{x^{q+1}}{q+1}\Big|_{x=a} = \frac{b^{q+1}-a^{q+1}}{q+1}\,.$$

Example 4.27. Since an anti-derivative of the sine function is negative of cosine, we find that

$$\int_a^b \sin x\,dx = (-\cos)(b) - (-\cos)(b) = \cos b - \cos a\,.$$

Theorem 4.28

Let $f : [a,b] \to \mathbb{R}$ be continuous and f is differentiable on (a,b). If f' is Riemann integrable on $[a,b]$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a)\,.$$

Remark 4.29. If f' is continuous on $[a,b]$, then the theorem above is simply a direct consequence of the Fundamental Theorem of Calculus. The theorem above can be viewed as a generalization of the Fundamental Theorem of Calculus.

Proof of Theorem 4.28. Let $\varepsilon > 0$ be given, and define $A = \int_a^b f'(x)\,dx$. By the definition of the integrability there exists $\delta > 0$ such that if $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a,b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums of f for $\mathcal{P}$ belongs to the interval $(A-\varepsilon, A+\varepsilon)$.

Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$ satisfying that $\|\mathcal{P}\| < \delta$. Then by the mean value theorem, for each $1 \leqslant i \leqslant n$ there exists $x_{i-1} < c < x_i$ such that $f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$. Since

$$\sum_{i=1}^{n} f'(c_i)(x_i - x_{i-1})$$

is a Riemann sum of f for $\mathcal{P}$, we must have

$$\Big|\sum_{i=1}^{n} f'(c_i)(x_i - x_{i-1}) - A\Big| < \varepsilon\,.$$

On the other hand, by the fact that

$$\begin{aligned}\sum_{i=1}^{n} f'(c_i)(x_i - x_{i-1}) &= \sum_{i=1}^{n} \big[f(x_i) - f(x_{i-1})\big] \\ &= f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1}) \\ &= f(x_n) - f(x_0) = f(b) - f(a)\,,\end{aligned}$$

we conclude that

$$\Big|f(b) - f(a) - \int_a^b f'(x)\,dx\Big| < \varepsilon\,.$$

Since $\varepsilon > 0$ is chosen arbitrarily, we find that $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$. □

Definition 4.30

An anti-derivative of f, if exists, is denoted by $\displaystyle\int f(x)\,dx$, and sometimes is also called an indefinite integral of f.