微積分 MA1001-A 上課筆記（精簡版）

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Definition 4.6: Partition of Intervals and Riemann Sums

A finite set $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is said to be a partition of the closed interval $[a, b]$ if $a = x_0 < x_1 < \cdots < x_n = b$. Such a partition $\mathcal{P}$ is usually denoted by $\{a = x_0 < x_1 < \cdots < x_n\}$. The norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is the number $\max\{x_i - x_{i-1} \,|\, 1 \leqslant i \leqslant n\}$; that is,

$$\|\mathcal{P}\| \equiv \max\left\{x_i - x_{i-1} \,\middle|\, 1 \leqslant i \leqslant n\right\}.$$

A partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is called regular if $x_i - x_{i-1} = \|\mathcal{P}\|$ for all $1 \leqslant i \leqslant n$.

Let $f : [a, b] \to \mathbb{R}$ be a function. A Riemann sum of f for the the partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ is a sum which takes the form

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}),$$

where the set $\Xi = \{c_0, c_1, \cdots, c_{n-1}\}$ satisfies that $x_{i-1} \leqslant c_i \leqslant x_i$ for each $1 \leqslant i \leqslant n$.

Definition 4.7: Riemann Integrals - 黎曼積分

Let $f : [a, b] \to \mathbb{R}$ be a function. f is said to be Riemann integrable on $[a, b]$ if there exists a real number A such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums for the partition $\mathcal{P}$ belongs to the interval $(A - \varepsilon, A + \varepsilon)$. Such a number A (is unique and) is called the Riemann integral of f on $[a, b]$ and is denoted by $\int_{[a,b]} f(x)\,dx$.

Remark 4.8. For conventional reason, the Riemann integral of f over the interval with left end-point a and right-end point b is written as $\int_a^b f(x)\,dx$, and is called the definite integral of f from a to b. The function f sometimes is called the integrand of the integral.

We also note that here in the representation of the integral, x is a dummy variable; that is, we can use any symbol to denote the independent variable; thus

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

and etc.

The following example shows that no all functions are Riemann integrable.

Example 4.9. Consider the Dirichlet function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational}\,, \\ 1 & \text{if } x \text{ is irrational}\,, \end{cases}$$

on the interval $[1, 2]$. By partitioning $[1, 2]$ into n sub-intervals with equal length, the Riemann sum given by the right end-point rule is always zero since the right end-point of each sub-interval is rational. On the other hand, by partitioning $[1, 2]$ into n sub-intervals using geometric sequence $1, r, r^2, \cdots, r^{n-1}, 2$, where $r = 2^{\frac{1}{n}}$, by the fact that $r^i \notin \mathbb{Q}$ for each $1 \leqslant i \leqslant n-1$ the Riemann sum of f for this partition given by the right end-point rule is

$$\begin{aligned} \sum_{i=1}^{n} f(r^i)(r^i - r^{i-1}) &= \sum_{i=1}^{n-1} (r^i - r^{i-1}) = r^1 - r^0 + r^2 - r^1 + \cdots + r^{n-1} - r^{n-2} \\ &= r^{n-1} - r^0 = \frac{2}{r} - 1 \end{aligned}$$

which approaches 1 as r approaches 1. Therefore, f is not integrable on $[1, 2]$ since there are two possible limits of Riemann sums which means that the Riemann sums cannot concentrate around any firxed real number.

Theorem 4.10

If $f : [a, b] \to \mathbb{R}$ is continuous, then f is Riemann integrable on $[a, b]$.

Example 4.11. In this example we compute $\displaystyle\int_a^b x^q\,dx$ when $q \neq -1$ is a rational number and $0 < a < b$. Since $f(x) = x^q$ is continuous on $[a, b]$, by Theorem 4.10 to find the integral it suffices to find the limit of the Riemann sum given by the left end-point rule as $\|\mathcal{P}\|$ approaches 0.

We follow the idea in Example 4.5. Let $r = \left(\dfrac{b}{a}\right)^{\frac{1}{n}}$ and $x_i = ar^i$, as well as the partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Then the Riemann sum of f for the partition $\mathcal{P}$ given by left end-point rule is

$$\begin{aligned} L(\mathcal{P}) &= \sum_{i=1}^{n} (ar^{i-1})^q (ar^i - ar^{i-1}) = a^{q+1}(r-1) \sum_{i=1}^{n} r^{(i-1)(q+1)} = a^{q+1}(r-1) \frac{r^{n(q+1)} - 1}{r^{q+1} - 1} \\ &= \frac{r-1}{r^{q+1} - 1} \big(b^{q+1} - a^{q+1}\big)\,. \end{aligned}$$

Since $\left.\dfrac{d}{dr}\right|_{r=1} r^{q+1} = (q+1)$, we have

$$\lim_{r\to 1} \frac{r^{q+1}-1}{r-1} = \left.\frac{d}{dr}\right|_{r=1} r^{q+1} = q+1\,;$$

thus by the fact that $r \to 1$ as $n \to \infty$ (or $\|\mathcal{P}\| \to 0$), we find that

$$\lim_{\|\mathcal{P}\|\to 0} L(\mathcal{P}) = \lim_{\|\mathcal{P}\|\to 0} L(\mathcal{P}) = \frac{b^{q+1}-a^{q+1}}{q+1}\,.$$

Therefore, $\displaystyle\int_a^b x^q\,dx = \frac{b^{q+1}-a^{q+1}}{q+1}$ if $q \neq 1$ is a rational number and $0 < a < b$.

Example 4.12. Since the sine function is continuous on any closed interval $[a,b]$, to find $\displaystyle\int_a^b \sin x\,dx$ we can partition $[a,b]$ into sub-intervals with equal length, use the right end-point rule to find an approximated value of the integral, and finally find the integral by passing the number of sub-intervals to the limit.

Let $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$. The right end-point rule gives the approximation

$$\sum_{i=1}^{n} \sin x_i \Delta x = \sum_{i=1}^{n} \sin(a+i\Delta x)\Delta x = \Delta x \sum_{i=1}^{n} \sin(a+i\Delta x)$$

of the integral.

Using the sum and difference formula, we find that

$$\cos\big[a + \big(i-\frac{1}{2}\big)\Delta x\big] - \cos\big[a + \big(i+\frac{1}{2}\big)\Delta x\big] = 2\sin(a+i\Delta x)\sin\frac{\Delta x}{2}\,;$$

thus if $\sin\dfrac{\Delta x}{2} \neq 0$,

$$\begin{aligned}\sum_{i=1}^{n} \sin(a+i\Delta x) = \frac{1}{2\sin\frac{\Delta x}{2}}\Big[\Big(\cos\big(a+\frac{1}{2}\Delta x\big) - \cos\big(a+\frac{3}{2}\Delta x\big)\Big) + \Big(\cos\big(a+\frac{3}{2}\Delta x\big) \\ - \cos\big(a+\frac{5}{2}\Delta x\big)\Big) + \cdots + \cos\big[a+(n-\frac{1}{2})\Delta x\big] \\ - \cos\big[a+\big(n+\frac{1}{2}\big)\Delta x\big]\Big]\end{aligned}$$

which, by the fact that $a + \big(n + \frac{1}{2}\Delta x\big) = b + \frac{1}{2}\Delta x$, implies that

$$\sum_{i=1}^{n} \sin x_i \Delta x = \frac{\frac{\Delta x}{2}}{\sin\frac{\Delta x}{2}}\Big[\cos\big(a+\frac{1}{2}\Delta x\big) - \cos\big(b+\frac{1}{2}\Delta x\big)\Big]\,.$$

By the fact that $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ and the continuity of the cosine function, we conclude that

$$\int_a^b \sin x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \sin x_i \Delta x = \cos a - \cos b \, .$$

Theorem 4.13

Let $f : [a, b] \to \mathbb{R}$ be a non-negative and continuous function. The area of the region enclosed by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is $\int_a^b f(x)\, dx$.

Example 4.14. In this example we use the integral notation to denote the areas of some common geometric figures (without really doing computations):

1. $\int_{-2}^{2} \sqrt{4 - x^2}\, dx = 2\pi$; 2. $\int_{-1}^{1} \sqrt{4 - x^2}\, dx = \dfrac{2\pi}{3} + \sqrt{3}$; 3. $\int_{-1}^{\sqrt{3}} \sqrt{4 - x^2}\, dx = \pi + \sqrt{3}$.

4.2.1 Properties of Definite Integrals

Definition 4.15

1. If f is defined at $x = a$, then $\int_a^a f(x)\, dx = 0$.
2. If f is integrable on $[a, b]$, then $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx = -\int_{[a,b]} f(x)\, dx$.

Remark 4.16. By the definition above, if f is Riemann integrable on $[a, b]$, $\int_b^a f(x)\, dx$ is the limit of the sum

$$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-}) \quad \textit{and} \quad \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

as $\max\big\{|x_i - x_{i-1}| \,\big|\, 1 \leqslant i \leqslant n\big\} \to 0$, where $x_0 = b > x_1 > x_2 > \cdots > x_n = a$.

Theorem 4.17

If f is Riemann integrable on the three closed intervals determined by a, b and c, then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx \, .$$

Theorem 4.18

Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and k be a constant. Then the function $kf \pm g$ are Riemann integrable on $[a, b]$, and

$$\int_a^b (kf \pm g)(x)\,dx = k\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx\,.$$

Theorem 4.19

If f is non-negative and Riemann integrable on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \geqslant 0$.

Corollary 4.20

If f, g are Riemann integrable on $[a, b]$ and $f(x) \leqslant g(x)$ for all $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx\,.$$

Theorem 4.21

If f is Riemann integrable on $[a, b]$, then $|f|$ is Riemann integrable on $[a, b]$ and

$$\Big|\int_a^b f(x)\,dx\Big| \leqslant \int_a^b \big|f(x)\big|\,dx\,.$$