微積分 MA1001-A 上課筆記（精簡版）

2018.10.23.

Ching-hsiao Arthur Cheng 鄭經斅

Theorem 3.2: Extreme Value Theorem - 極值定理

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.（連續函數在閉區間上必有最大最小值）

Definition 3.4

Let f be defined on an open interval containing c. The number/point c is called a critical number or critical point of f if $f'(c) = 0$ or if f is not differentiable at c.

Theorem 3.5

If f has a relative minimum or relative maximum at $x = c$, then c is a critical point of f.

The way to find extrema of a continuous function f on a closed interval $[a, b]$:

1. Find the critical points of f in (a, b).
2. Evaluate f at each critical points in (a, b).
3. Evaluate f at the end-points of $[a, b]$.
4. The least of these values is the minimum, and the greatest is the maximum.

Theorem 3.7: Rolle's Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and f is differentiable on (a, b). If $f(a) = f(b)$, then there is at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Theorem 3.8: Mean Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable on (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

We also prove, by the Mean Value Theorem, that

$$|\sin x - \sin y| \leqslant |x - y| \qquad \forall\, x, y \in \mathbb{R}\,,$$
$$|\cos x - \cos y| \leqslant |x - y| \qquad \forall\, x, y \in \mathbb{R}\,.$$

3.3 Monotone Functions and the First Derivative Test

Definition 3.11

Let f be defined on an interval I.

1. f is said to be increasing on I if
$$f(x_1) \leqslant f(x_2) \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 < x_2\,.$$

2. f is said to be decreasing on I if
$$f(x_1) \geqslant f(x_2) \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 < x_2\,.$$

3. f is said to be strictly increasing on I if
$$f(x_1) < f(x_2) \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 < x_2\,.$$

4. f is said to be strictly decreasing on I if
$$f(x_1) > f(x_2) \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 < x_2\,.$$

When f is either increasing on I or decreasing on I, then f is said to be monotone. When f is either strictly increasing on I or strictly decreasing on I, then f is said to be strictly monotone on I.

Remark 3.12. Note that f is increasing on I if
$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} \geqslant 0 \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 \neq x_2\,.$$
Therefore, f is increasing on I if the slope of each secant line of the graph of f is non-negative. Similar conclusions hold for the other cases.

Example 3.13. The function $f(x) = x^3$ is strictly increasing on $\mathbb{R}$, and $f(x) = -x^3$ is strictly decreasing on $\mathbb{R}$.

Example 3.14. The sine function is strictly increasing on $\big[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\big]$ for all $n \in \mathbb{Z}$, but decreasing on $\big[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\big]$ for all $n \in \mathbb{Z}$. However, the sine function is **not** strictly increasing on $\bigcup\limits_{n=-\infty}^{\infty} \big[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\big]$ and is **not** strictly decreasing on $\bigcup\limits_{n=-\infty}^{\infty} \big[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\big]$.

Theorem 3.15

Let $f : [a, b] \to \mathbb{R}$ be continuous and f is differentiable on (a, b).

1. If $f'(x) \geqslant 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.
2. If $f'(x) \leqslant 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.
3. If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on $[a, b]$.
4. If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on $[a, b]$.

Proof. We only prove 1 since all the other conclusion can be proved in a similar fashion.

Suppose that $f'(x) \geqslant 0$, and $x_1 < x_2$. By the Mean Value Theorem, there exists $c \in (x_1, x_2)$ such that

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(c) \geqslant 0\,;$$

thus $f(x_1) \leqslant f(x_2)$ if $x_1 < x_2$. □

Remark 3.16. The condition $f'(x) > 0$ is just a sufficient condition for that f is strictly increasing, but not a necessary condition. For example, $f(x) = x^3$ is strictly increasing on $\mathbb{R}$, but $f'(0) = 0$.

Theorem 3.17: The First Derivative Test

Let f be a continuous function defined on an open interval I containing c. If f is differentiable on I, except possibly at c, then

1. If f' changes from negative to positive at c, then $f(c)$ is a local minimum of f.
2. If f' changes from positive to negative at c, then $f(c)$ is a local maximum of f.
3. If f' is sign definite on $I\backslash\{c\}$, then $f(c)$ is neither a relative minimum or relative maximum of f.

Proof. We only prove 1. Assume that f' changes from negative to positive at c. Then there exists a and b in I such that

$$f'(x) < 0 \text{ for all } x \in (a, c) \quad \text{and} \quad f'(x) > 0 \text{ for all } x \in (c, b)\,.$$

By Theorem 3.15, f is decreasing on (a, c) and is increasing on (c, b). Therefore, $f(c)$ is a minimum on an open interval (a, b); thus is a relative minimum on I. □

Example 3.18. Find the relative extrema of $f(x) = \frac{1}{2}x - \sin x$ in the interval $(0, 2\pi)$.

By Theorem 3.5 the relative extrema occurs at critical points. Since f is differentiable on $(0, 2\pi)$, a critical point x satisfies

$$0 = f'(x) = \frac{1}{2} - \cos x$$

which implies that $c = \frac{\pi}{3}$ and $c = \frac{5\pi}{3}$ are the only critical points. To determine if $f\left(\frac{\pi}{3}\right)$ or $f\left(\frac{5\pi}{3}\right)$ is a relative minimum, we apply Theorem 3.17 and found that, since f' changes from negative to positive at $\frac{\pi}{3}$ and changes from positive to negative at $\frac{5\pi}{3}$, $f\left(\frac{\pi}{3}\right)$ is a relative minimum of f on $(0, 2\pi)$.

3.4 Concavity（凹性）and the Second Derivative Test

Definition 3.19

Let f be differentiable on an open interval I. The graph of f is concave upward（凹向上）on I if f' is strictly increasing on the interval and concave downward（凹向下）on I if f' is strictly decreasing on the interval.

Remark 3.20. It does not really matter if f' has to be strictly monotone, instead of just monotone, in order to define the concavity of the graph of f. Here we define the concavity by the strict monotonicity.

- Graphical interpretation of concavity: Let f be differentiable on an open interval I.
  1. If the graph of f is concave upward on I, then the graph of f lies above all of its tangent lines on I.
  2. If the graph of f is concave downward on I, then the graph of f lies below all of its tangent lines on I.

The following theorem is a direct consequence of Theorem 3.15.

Theorem 3.21: Test for Concavity

Let f be a twice differentiable function on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example 3.22. Determine the open intervals on which the graph of $f(x) = \dfrac{6}{x^2+3}$ is concave upward or concave downward.

First we compute the second derivative of f:

$$f'(x) = \frac{-12x}{(x^2+3)^2} \Rightarrow f''(x) = -12\frac{(x^2+3)^2 - 2(x^2+3)(2x)x}{(x^2+3)^4} = \frac{36(x^2-1)}{(x^2+3)^3}.$$

Therefore, the graph of f is concave upward if $x > 1$ and is concave downward if $x < 1$.

Definition 3.23: Point of inflection（反曲點）

Let f be a differentiable function on an open interval containing c. The point $\big(c, f(c)\big)$ is called a point of inflection (or simply an inflection point) of the graph of f if the concavity of f changes from upward to downward or downward to upward at this point.