# 微積分 MA1001-A 上課筆記（精簡版）

**2018.10.18.**

**Ching-hsiao Arthur Cheng** 鄭經斅

## 3.1 Extrema on an Interval

**Definition 3.1**

Let $f$ be defined on an interval $I$ containing $c$.

1. $f(c)$ is the (absolute, global) minimum of $f$ on $I$ when $f(c) \leqslant f(x)\ \forall\, x \in I$.

2. $f(c)$ is the (absolute, global) maximum of $f$ on $I$ when $f(c) \geqslant f(x)\ \forall\, x \in I$.

The minimum and maximum of a function on an interval are the extreme values, or extrema (the singular form of extrema is extremum), of the function on the interval. Extrema that occur at the end-points are called end-point extrema.

**Theorem 3.2: Extreme Value Theorem - 極值定理**

If $f$ is continuous on a closed interval $[a, b]$, then $f$ has both a minimum and a maximum on the interval.（連續函數在閉區間上必有最大最小值）

**Definition 3.3**

Let $f$ be defined on an interval $I$ containing $c$.

1. If there is an open interval containing $c$ on which $f(c)$ is a maximum, then $f(c)$ is called a relative/local maximum of $f$.

2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a relative/local minimum of $f$.

**Definition 3.4**

Let $f$ be defined on an open interval containing $c$. The number/point $c$ is called a critical number or critical point of $f$ if $f'(c) = 0$ or if $f$ is not differentiable at $c$.

**Theorem 3.5**

If $f$ has a relative minimum or relative maximum at $x = c$, then $c$ is a critical point of $f$.

*Proof.* W.L.O.G., we assume that $f$ is differentiable at $c$. If $f'(c) > 0$, then there exists $\delta_1 > 0$ such that

$$\left|\frac{f(x) - f(c)}{x - c} - f'(c)\right| < \frac{f'(c)}{2} \qquad \text{if} \quad 0 < |x - c| < \delta_1\,;$$

thus

$$\frac{f'(c)}{2} < \frac{f(x)-f(c)}{x-c} < \frac{3f'(c)}{2} \qquad \text{if} \quad 0 < |x-c| < \delta_1\,.$$

1. If $0 < x - c < \delta_1$,

$$f(c) + \frac{f'(c)}{2}(x-c) < f(x) < f(c) + \frac{3f'(c)}{2}(x-c)$$

   which implies that $f$ cannot attain a relative maximum at $x = c$ since $f(x) > f(c)$ on the right-hand side of $c$.

2. if $-\delta < x - c < 0$,

$$f(c) + \frac{f'(c)}{2}(x-c) > f(x) > f(c) + \frac{3f'(c)}{2}(x-c)$$

   which implies that $f$ cannot attain a relative minimum at $x = c$ since $f(c) > f(x)$ on the left-hand side of $c$.

Therefore, we conclude that if $f'(c) > 0$, then $f$ cannot attain either a relative maximum or minimum at $x = c$. Similar conclusion can be drawn for the case $f'(c) < 0$; thus if $f$ attains a relative extremum at $x = c$, then $f'(c) = 0$. □

**The way to find extrema of a continuous function $f$ on a closed interval** $[a, b]$:

1. Find the critical points of $f$ in $(a, b)$.
2. Evaluate $f$ at each critical points in $(a, b)$.
3. Evaluate $f$ at the end-points of $[a, b]$.
4. The least of these values is the minimum, and the greatest is the maximum.

**Example 3.6.** Find the extrema of $f(x) = 2\sin x - \cos 2x$ on the interval $[0, 2\pi]$.

Since $f$ is differentiable on $(0, 2\pi)$, a critical point $c$ satisfies

$$0 = f'(c) = 2\cos c + 2\sin 2c = 2\cos c(1 + 2\sin c)\,.$$

Therefore, $c = \frac{\pi}{2}$, $c = \frac{3\pi}{2}$, $c = \frac{7\pi}{6}$ or $c = \frac{11\pi}{6}$, and the values of $f$ at these critical points are

$$f\big(\frac{\pi}{2}\big) = 2\cdot 1 - (-1) = 3\,, \qquad f\big(\frac{3\pi}{2}\big) = 2\cdot(-1) - (-1) = -1\,,$$
$$f\big(\frac{7\pi}{6}\big) = 2\cdot\big(-\frac{1}{2}\big) - \frac{1}{2} = -\frac{3}{2}\,, \qquad f\big(\frac{11\pi}{6}\big) = 2\cdot\big(-\frac{1}{2}\big) - \frac{1}{2} = -\frac{3}{2}\,.$$

On the other hand, the values of $f$ at the end-points are

$$f(0) = 2 \cdot 0 - 1 = -1 \quad \text{and} \quad f(2\pi) = 2 \cdot 0 - 1 = -1\,.$$

Therefore, $f\left(\frac{\pi}{2}\right) = 3$ is the maximum of $f$ on $[0, 2\pi]$, while the minimum of $f$ on $[0, 2\pi]$ occurs at $c = \frac{7\pi}{6}$ and $c = \frac{11\pi}{6}$ and the minimum is $-\frac{3}{2}$.

## 3.2 Rolle's Theorem and the Mean Value Theorem

**Theorem 3.7: Rolle's Theorem**

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $f$ is differentiable on $(a, b)$. If $f(a) = f(b)$, then there is at least one point $c \in (a, b)$ such that $f'(c) = 0$.

*Proof.* If $f$ is a constant function, then $f'(x) = 0$ for all $x \in (a, b)$. Now suppose that $f$ is not a constant function on $[a, b]$, by the Extreme Value Theorem implies that $f$ has a maximum and a minimum on $[a, b]$, and the maximum and the minimum of $f$ on $[a, b]$ are different. Therefore, there must be a point $c \in (a, b)$ at which $f$ attains its extreme value. By Theorem 3.5, $f'(c) = 0$. □

**Theorem 3.8: Mean Value Theorem**

If $f : [a, b] \to \mathbb{R}$ is continuous and $f$ is differentiable on $(a, b)$, then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

*Proof.* Define $g : [a, b] \to \mathbb{R}$ by $g(x) = \big[f(x) - f(a)\big](b - a) - \big[f(b) - f(a)\big](x - a)$. Then $g : [a, b] \to \mathbb{R}$ is continuous and $g$ is differentiable on $(a, b)$. Moreover, $g(a) = g(b) = 0$; thus the Rolle Theorem implies that there exists $c \in (a, b)$ such that $g'(c) = 0$. On the other hand,

$$0 = g'(c) = (b - a)f'(c) - \big[f(b) - f(a)\big]\,;$$

thus there exists $c \in (a, b)$ satisfying $f'(c) = \frac{f(b) - f(a)}{b - a}$. □

**Remark 3.9.** In fact, by modifying the proof of the mean value theorem a little bit, we can show the following: If $f, g : [a, b] \to \mathbb{R}$ are continuous, $f, g$ are differentiable on $(a, b)$ and $g(a) \neq g(b)$, then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} .$$

The statement above is a generalization of the mean value theorem and is called the Cauchy mean value theorem.

**Example 3.10.** Note that the sine function is continuous on any closed interval $[a, b]$ and is differentiable on $(a, b)$. Therefore, the mean value theorem implies that there exists $c \in (a, b)$ such that

$$\cos c = \frac{d}{dx}\Big|_{x=c} \sin x = \frac{\sin b - \sin a}{b - a}$$

which implies that $|\sin a - \sin b| = |\cos c||a - b| \leqslant |a - b|$. Therefore,

$$|\sin x - \sin y| \leqslant |x - y| \qquad \forall\, x, y \in \mathbb{R} .$$

Similarly,

$$|\cos x - \cos y| \leqslant |x - y| \qquad \forall\, x, y \in \mathbb{R} .$$