微積分 MA1001-A 上課筆記（精簡版）

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Definition 2.2

Let f be a function defined on an open interval I containing c. f is said to be differentiable at c if the limit

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

exists. If the limit above exists, the limit is denoted by $f'(c)$ and called the derivative of f at c. When the derivative of f at each point of I exists, f is said to be differentiable on I and the derivative of f is a function denoted by f'.

Theorem 2.9: 可微必連續

Let f be a function defined on an open interval I, and $c \in I$. If f is differentiable at c, then f is continuous at c.

Theorem 2.11

We have the following differentiation rules:

1. If n is an integer, then $\dfrac{d}{dx}x^n = nx^{n-1}$ (whenever x^{n-1} makes sense or $x^n \in \mathbb{R}$).
2. $\dfrac{d}{dx}\sin x = \cos x$, $\dfrac{d}{dx}\cos x = -\sin x$.
3. If k is a constant and $f : (a,b) \to \mathbb{R}$ is differentiable at $c \in (a,b)$, then kf is differentiable at c and
$$\frac{d}{dx}\Big|_{x=c}\big[kf(x)\big] = kf'(c)\,.$$
4. If $f, g : (a,b) \to \mathbb{R}$ are differentiable at $c \in (a,b)$, then $f \pm g$ is differentiable at c and
$$\frac{d}{dx}\Big|_{x=c}\big[f(x) \pm g(x)\big] = f'(c) \pm g'(c)\,.$$

Theorem 2.13: Product Rule

Let $f, g : (a,b) \to \mathbb{R}$ be real-valued functions, and $c \in (a,b)$. If f and g are differentiable at c, then fg is differentiable at c and

$$\frac{d}{dx}\Big|_{x=c}(fg)(x) = f'(c)g(c) + f(c)g'(c)\,.$$

Theorem 2.15: Quotient Rule

Let $f, g : (a,b) \to \mathbb{R}$ be real-valued functions, and $c \in (a,b)$. If f and g are differentiable at c and $g(c) \neq 0$, then $\dfrac{f}{g}$ is differentiable at c and

$$\frac{d}{dx}\Big|_{x=c} \frac{f}{g}(x) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

We also used the quotient rule to show the following identities:

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\cot x = -\csc^2 x,$$
$$\frac{d}{dx}\sec x = \sec x \tan x, \qquad \frac{d}{dx}\csc x = -\csc x \cot x.$$

2.3 The Chain Rule

The chain rule is used to study the derivative of composite functions.

Theorem 2.18: Chain Rule - 連鎖律

Let I, J be open intervals, $f : J \to \mathbb{R}$, $g : I \to \mathbb{R}$ be real-valued functions, and the range of g is contained in J. If g is differentiable at $c \in I$ and f is differentiable at $g(c)$, then $f \circ g$ is differentiable at c and

$$\frac{d}{dx}\Big|_{x=c} (f \circ g)(x) = f'(g(c))g'(c).$$

Proof. To simplify the notation, we set $d = g(c)$.

Let $\varepsilon > 0$ be given. Since f is differentiable at d and g is differentiable at c, there exist $\delta_1, \delta_2 > 0$ such that

$$\Big|\frac{f(d+k) - f(d)}{k} - f'(d)\Big| < \frac{\varepsilon}{2(1 + |g'(c)|)} \quad \text{if} \quad 0 < |k| < \delta_1,$$
$$\Big|\frac{g(c+h) - g(c)}{h} - g'(c)\Big| < \min\Big\{1, \frac{\varepsilon}{2(1 + |f'(d)|)}\Big\} \quad \text{if} \quad 0 < |h| < \delta_2.$$

Therefore,

$$\big|f(d+k) - f(d) - f'(d)k\big| \leqslant \frac{\varepsilon}{2(1 + |g'(c)|)}|k| \quad \text{if} \quad |k| < \delta_1,$$
$$\big|g(c+h) - g(c) - g'(c)h\big| \leqslant \min\Big\{1, \frac{\varepsilon}{2(1 + |f'(d)|)}\Big\}|h| \quad \text{if} \quad |h| < \delta_2.$$

By Theorem 2.9, g is continuous at c; thus $\lim\limits_{h\to 0} g(c+h) = g(c)$. This fact provides $\delta_3 > 0$ such that

$$\big|g(c+h) - g(c)\big| < \delta_1 \quad \text{if} \quad |h| < \delta_3 \,.$$

Define $\delta = \min\{\delta_2, \delta_3\}$. Then $\delta > 0$. Moreover, if $|h| < \delta$, the number $k \equiv g(c+h) - g(c)$ satisfies $|k| < \delta_1$. As a consequence, if $|h| < \delta$,

$$\begin{aligned}
\big|(f\circ g)(c+h) - (f\circ g)(c) - f'(d)g'(c)h\big| &= \big|f(g(c+h)) - f(d) - f'(d)g'(c)h\big| \\
&= \big|f(d+k) - f(d) - f'(d)g'(c)h\big| \\
&= \big|f(d+k) - f(d) - f'(d)k + f'(d)k - f'(d)g'(c)h\big| \\
&\leqslant \big|f(d+k) - f(d) - f'(d)k\big| + \big|f'(d)\big|\big|k - g'(c)h\big| \\
&\leqslant \frac{\varepsilon}{2(1+|g'(c)|)}|k| + \big|f'(d)\big|\big|g(c+h) - g(c) - g'(c)h\big| \\
&\leqslant \frac{\varepsilon}{2(1+|g'(c)|)}\big(|k - g'(c)h| + |g'(c)||h|\big) + \big|f'(d)\big|\frac{\varepsilon}{2(1+|f'(d)|)} \\
&\leqslant \frac{\varepsilon}{2(1+|g'(c)|)}\big(|h| + |g'(c)||h|\big) + \big|f'(d)\big|\frac{\varepsilon|h|}{2(1+|f'(d)|)} \\
&= \frac{\varepsilon}{2}|h| + \frac{\big|f'(d)\big|}{2(1+|f'(d)|)}\varepsilon|h| \,.
\end{aligned}$$

The inequality above implies that if $0 < |h| < \delta$,

$$\Big|\frac{(f\circ g)(c+h) - (f\circ g)(c)}{h} - f'(d)g'(c)\Big| \leqslant \frac{\varepsilon}{2} + \frac{\big|f'(d)\big|}{2(1+|f'(d)|)}\varepsilon < \varepsilon$$

which concludes the chain rule. □

How to memorize the chain rule? Let $y = g(x)$ and $u = f(y)$. Then the derivative $u = (f\circ g)(x)$ is $\dfrac{du}{dx} = \dfrac{du}{dy}\dfrac{dy}{dx}$.

Example 2.19. Let $f(x) = (3x - 2x^2)^3$. Then $f'(x) = 3(3x - 2x^2)^2(3 - 4x)$.

Example 2.20. Let $f(x) = \Big(\dfrac{3x-1}{x^2+3}\Big)^2$. Then

$$\begin{aligned}
f'(x) &= 2\Big(\frac{3x-1}{x^2+3}\Big)^{2-1}\frac{d}{dx}\frac{3x-1}{x^2+3} = \frac{2(3x-1)}{x^2+3}\cdot\frac{3(x^2+3) - 2x(3x-1)}{(x^2+3)^2} \\
&= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3} \,.
\end{aligned}$$

Example 2.21. Let $f(x) = \tan^3\left[(x^2-1)^2\right]$. Then

$$\begin{aligned} f'(x) &= \left\{3\tan^2\left[(x^2-1)^2\right]\sec^2\left[(x^2-1)^2\right]\right\} \times \left[2(x^2-1)\cdot(2x)\right] \\ &= 12x(x^2-1)\tan^2\left[(x^2-1)^2\right]\sec^2\left[(x^2-1)^2\right]. \end{aligned}$$

Example 2.22. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 . \end{cases}$$

Then if $x \neq 0$, by the chain rule we have

$$\begin{aligned} f'(x) &= \Big(\frac{d}{dx}x^2\Big)\sin\frac{1}{x} + x^2\Big(\frac{d}{dx}\sin\frac{1}{x}\Big) = 2x\sin\frac{1}{x} + x^2\cos\frac{1}{x}\Big(\frac{d}{dx}\frac{1}{x}\Big) \\ &= 2x\sin\frac{1}{x} + x^2\cos\frac{1}{x}\Big(-\frac{1}{x^2}\Big) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}. \end{aligned}$$

Next we compute $f'(0)$. If $\Delta x \neq 0$, we have

$$\Big|\frac{f(\Delta x) - f(0)}{\Delta x}\Big| = \Big|\Delta x \sin\frac{1}{\Delta x}\Big| \leqslant |\Delta x|;$$

thus $-|\Delta x| \leqslant \dfrac{f(\Delta x) - f(0)}{\Delta x} \leqslant |\Delta x|$ for all $\Delta x \neq 0$ and the Squeeze Theorem implies that

$$f'(0) = \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x} = 0.$$

Therefore, we conclude that

$$f'(x) = \begin{cases} 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 . \end{cases}$$

Definition 2.23

Let f be a function defined on an open interval I. f is said to be continuously differentiable on I if f is differentiable on I and f' is continuous on I.

The function f given in Example 2.22 is differentiable on $\mathbb{R}$ but not continuously differentiable since $\lim\limits_{x \to 0} f'(x)$ D.N.E.