微積分 MA1001-A 上課筆記（精簡版）

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1.3 Continuity of Functions

Definition 1.34

Let f be a function defined on an interval I, and $c \in I$.

1. f is said to be right-continuous at c (or continuous from the right at c) if
$$\lim_{x \to c^+} f(x) = f(c) .$$
2. f is said to be left-continuous at c (or continuous from the left at c) if
$$\lim_{x \to c^-} f(x) = f(c) .$$
3. If c is the left end-point of I, f is said to be continuous at c if f is right-continuous at c.
4. If c is the right end-point of I, f is said to be continuous at c if f is left-continuous at c.
5. If c is an interior point of I; that is, c is neither the left end-point nor the right end-point of I, then f is said to be continuous at c if $\lim\limits_{x \to c} f(x) = f(c)$.

f is said to be discontinuous at c if f is not continuous at c, and in this case c is called a point of discontinuity (or simply a discontinuity) of f. f is said to be continuous (or a continuous function) on I if f is continuous at each point of I.

Example 1.35. Consider the the greatest integer function (also known as the Gauss function or the floor function) $[\![\cdot]\!] : \mathbb{R} \to \mathbb{R}$ defined by

$$[\![x]\!] = \text{the greatest integer which is not greater than } x.$$

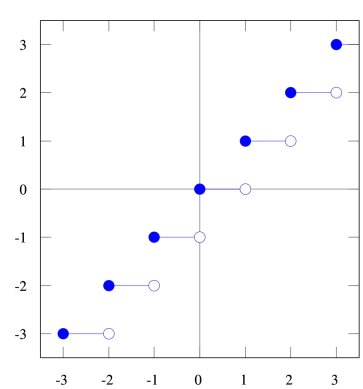

Figure 1.8: The greatest integer function $y = [\![x]\!]$

For example, $[\![2.5]\!] = 2$ and $[\![-2.5]\!] = -3$. If c is not an integer, $\lim\limits_{x\to c} [\![x]\!] = c$, while if c is an integer, we have

$$\lim_{x\to c^+} [\![x]\!] = c \qquad \text{and} \qquad \lim_{x\to c^-} [\![x]\!] = c - 1\,.$$

Let $f : [0,2] \to \mathbb{R}$ be given by $f(x) = [\![x]\!]$. Then the conclusion above shows that f is continuous at every non-integer number, while f is not continuous at 1 (since $\lim\limits_{x\to 1} f(x)$ does not exist) and 2 (since $\lim\limits_{x\to 2^-} f(x) \neq f(2)$). On the other hand, $\lim\limits_{x\to 0^+} f(x) = f(0)$, so f is continuous at 0.

Therefore, f is continuous at c if c is not an integer, and f is right-continuous at c if c is an integer.

Example 1.36. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}\,, \\ -x & \text{if } x \notin \mathbb{Q}\,. \end{cases}$$

By the fact that $|f(x)| \leqslant |x|$ for all $x \in \mathbb{R}$, we find that $-|x| \leqslant f(x) \leqslant |x|$ for all $x \in \mathbb{R}$. By the Squeeze Theorem, $\lim\limits_{x\to 0} f(x) = 0 = f(0)$; thus f is continuous at 0. Is f continuous at other numbers?

Example 1.37. Recall the Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ in Example 1.5 given by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}\,, \\ 1 & \text{if } x \notin \mathbb{Q}\,, \end{cases}$$

We have explained (but not proven) that the limit $\lim\limits_{x\to c} f(x)$ does not exist for all $c \in (0,\infty)$; thus f is discontinuous at all real numbers.

Example 1.38. Let $f(x) = x^n$, where n is a positive integer. We have shown that $\lim\limits_{x\to c} x^n = c^n$ for all real numbers c; thus f is continuous on $\mathbb{R}$.

Example 1.39. Recall the function $f : (0,\infty) \to \mathbb{R}$ in Example 1.6 given by

$$f(x) = \begin{cases} \dfrac{1}{p} & \text{if } x = \dfrac{q}{p}, \text{ where } p, q \in \mathbb{N} \text{ and } (p,q) = 1\,, \\ 0 & \text{if } x \text{ is irrational}\,. \end{cases}$$

We have explained (but not proven) that $\lim\limits_{x\to c} f(x) = 0$ for all $c \in (0,\infty)$. Therefore, f is continuous at all irrational numbers but is discontinuous at all rational numbers.

Remark 1.40. Let I be an interval, $c \in I$, and $f : I \to \mathbb{R}$ be a function. The continuity of f at c is equivalent to that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \text{if} \quad |x - c| < \delta \text{ and } x \in I\,.$$

To see this, we first consider the case that c is an interior point of I. Then by the definition, f is continuous at c if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \text{if} \quad 0 < |x - c| < \delta\,.$$

Since $|f(x) - f(c)| < \varepsilon$ automatically holds if $|x - c| = 0$, the statement above is equivalent to that

$$|f(x) - f(c)| < \varepsilon \quad \text{if} \quad |x - c| < \delta\,.$$

Now let us look at the case when c is the left end-point of I (so in this case $c \in I$). Then by definition, f is continuous at c if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \text{if} \quad 0 < x - c < \delta\,.$$

Again $|f(x) - f(c)| < \varepsilon$ automatically holds if $x - c = 0$, the statement above is equivalent to that

$$|f(x) - f(c)| < \varepsilon \quad \text{if} \quad c \leqslant x < c + \delta\,.$$

Note that since c is the left end-point, the set $\{x \,|\, c \leqslant x < c+\delta\}$ is the same as $\{x \,|\, |x-c| < \delta, x \in I\}$; thus the statement above is equivalent to that

$$|f(x) - f(c)| < \varepsilon \quad \text{if } |x - c| < \delta \text{ and } x \in I\,.$$

Similar argument can be applied to the case when c is the right end-point of I.

Remark 1.41. Discontinuities of functions can be classified into different categories: removable discontinuities and non-removable discontinuities. Let c be a discontinuity of a function f. Then either (1) $\lim\limits_{x\to c} f(x)$ exists but $\lim\limits_{x\to c} f(x) \neq f(c)$ or (2) $\lim\limits_{x\to c} f(x)$ does not exist. If it is the first case, then c is called a removable discontinuity and that means we can adjust/re-define the value of f at c to make it continuous at c. For the second case, no matter what $f(c)$ is, f cannot be continuous at c.

If $\lim\limits_{x\to c^+} f(x)$ and $\lim\limits_{x\to c^-} f(x)$ both exist but are not identical, c is also called a jump discontinuity.

Proposition 1.42

Let f, g be defined on an interval I, $c \in I$, and f, g be continuous at c. Then

1. $f \pm g$ is continuous at c.
2. fg is continuous at c.
3. $\frac{f}{g}$ is continuous at c if $g(c) \neq 0$.

Corollary 1.43

Let f, g be continuous functions on an interval I. Then

1. $f \pm g$ is continuous on I.
2. fg is continuous on I
3. $\frac{f}{g}$ is continuous (on its domain).