微積分 MA1001-A 上課筆記（精簡版）

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Definition 1.7

Let f be a function defined on an open interval containing c (except possibly at c), and L be a real number. The statement

$$\lim_{x\to c} f(x) = L\,, \quad \text{read "the limit of } f \text{ at } c \text{ is } L\text{"},$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \;\text{ if }\; 0 < |x - c| < \delta.$$

Theorem 1.12

Let b, c be real numbers, f, g be functions with $\lim\limits_{x\to c} f(x) = L$, $\lim\limits_{x\to c} g(x) = K$. Then

1. $\lim\limits_{x\to c} b = b$, $\lim\limits_{x\to c} x = c$, $\lim\limits_{x\to c} |x| = |c|$;
2. $\lim\limits_{x\to c} \big[f(x) \pm g(x)\big] = L + K$;（和或差的極限等於極限的和或差）
3. $\lim\limits_{x\to c} \big[f(x)g(x)\big] = LK$;（乘積的極限等於極限的乘積）
4. $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$ if $K \neq 0$.（若分母極限不為零，則商的極限等於極限的商）

Theorem 1.15

If $c > 0$ and n is a positive integer, then $\lim\limits_{x\to c} x^{\frac{1}{n}} = c^{\frac{1}{n}}$.

Theorem 1.16

If f and g are functions such that $\lim\limits_{x\to c} g(x) = K$, $\lim\limits_{x\to K} f(x) = L$ and $L = f(K)$, then

$$\lim_{x\to c} (f \circ g)(x) = L\,.$$

Theorem 1.18: Squeeze Theorem（夾擠定理）

Let f, g, h be functions defined on an interval containing c (except possibly at c), and $h(x) \leqslant f(x) \leqslant g(x)$ if $x \neq c$. If $\lim\limits_{x\to c} h(x) = \lim\limits_{x\to c} g(x) = L$, then $\lim\limits_{x\to c} f(x)$ exists and is equal to L.

Definition 1.23: One-sided limits

Let f be a function defined on an interval with c as the left/right end-point, and L be a real number. The statement

$$\lim_{x\to c^+} f(x) = L \Big/ \lim_{x\to c^-} f(x) = L\,,$$

read "the right/left(-hand) limit of f at c is L" or "the limit of f at c from the right/left is L", means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{if} \quad 0 < (x - c) < \delta \ / \ -\delta < x - c < 0.$$

We note that Theorem 1.12, Corollary 1.14, Theorem 1.15, 1.16 and 1.18 are also valid when the limits are replaced by one-sided limits. Theorem 1.16 is also valid when $x \to c$ is replaced by $x \to c^+$ or $x \to c^-$ (with $x \to K$ unchanged).

Theorem 1.25

Let f be a function defined on an open interval containing c (except possibly at c). The limit $\lim\limits_{x\to c} f(x)$ exists if and only if $\lim\limits_{x\to c^+} f(x)$ and $\lim\limits_{x\to c^-} f(x)$ both exist and are identical. In either case,

$$\lim_{x\to c} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)\,.$$

We also established the inequality $|\sin x| \leqslant |x|$ for all $x \in \mathbb{R}$ and $\sin x \leqslant x \leqslant \tan x$ if $0 < x < \dfrac{\pi}{2}$. Using these inequalities and Theorem 1.25, we conclude that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

Remark 1.27. The function $\dfrac{\sin x}{x}$ is the famous (unnormalized) sinc function; that is, $\operatorname{sinc}(x) = \dfrac{\sin x}{x}$ and $\operatorname{sinc}(0) = 1$. The example above shows that $\lim\limits_{x\to 0} \operatorname{sinc}(x) = \operatorname{sinc}(0)$.

Example 1.28. In this example we compute the limit $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$. By the half-angle formula, $1 - \cos x = 2\sin^2 \dfrac{x}{2}$; thus

$$\frac{1-\cos x}{x^2} = \frac{2\sin^2 \frac{x}{2}}{x^2} = \frac{1}{2}\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2} = \frac{1}{2}\operatorname{sinc}^2\left(\frac{x}{2}\right).$$

Therefore, Theorem 1.16 implies that $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$.

Explanation on "A if and only if B" in Theorem 1.25: It should be clear that "A if B" means "A happens when B happens" (which is the same as "B implies A"). The statement "A only if B" means that "A happens only when B happens"; thus "A only if B" means that "A implies B".

Proof of Theorem 1.25. (⇒) - the "only if" part: Suppose that $\lim_{x\to c} f(x) = L$, and let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{if} \quad 0 < |x - c| < \delta \,.$$

Therefore, there exists $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{if} \quad 0 < x - c < \delta \,;$$

thus $\lim_{x\to c^+} f(x) = L$. Similarly, $\lim_{x\to c^-} f(x) = L$.

(⇐) - the "if" part: Suppose that $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$. Let $\varepsilon > 0$. Then there exist $\delta_1, \delta_2 > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{if} \quad 0 < x - c < \delta_1$$

and

$$|f(x) - L| < \varepsilon \quad \text{if} \quad -\delta_2 < x - c < 0 \,.$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and if $0 < |x - c| < \delta$, we must have $0 < x - c < \delta_1$ and $-\delta_2 < x - c < 0$; thus if $0 < |x - c| < \delta$, we must have $|f(x) - L| < \varepsilon$. □

An open interval in the real number system can be unbounded. When the open interval on which f is defined is not bounded from above (which means there is no real number which is larger than all the numbers in this interval), we can also consider the behavior of $f(x)$ as x becomes increasingly large and eventually outgrow all finite bounds.

Definition 1.29: Limits as $x \to \pm\infty$

Let f be a function defined on an infinite interval bounded from below/above, and L be a real number. The statement

$$\lim_{x\to\infty} f(x) = L \Big/ \lim_{x\to-\infty} f(x) = L \,,$$

read "the right/left(-hand) limit of f at c is L" or "the limit of f at c from the right/left is L", means that for each $\varepsilon > 0$ there exists a real number $M > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{if} \quad x > M \;/\; x < -M.$$

Similar to the case of one-sided limit, Theorem 1.12, Corollary 1.14, Theorem 1.15, 1.16 and 1.18 are also valid when the $x \to c^{\pm}$ are replaced by $x \to \pm\infty$.

Example 1.30. In this example we show that $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to-\infty} f(x) = 0$.

Let $\varepsilon > 0$ be given. Define $M = \dfrac{1}{\varepsilon}$. Then if $x > M$ or $x < -M$, we must have $|x| > M$; thus if $x > M$ or $x < -M$,

$$\left|\frac{1}{x} - 0\right| = \frac{1}{|x|} < \frac{1}{M} < \varepsilon\,.$$

Similarly, $\lim\limits_{x\to\infty} \dfrac{1}{|x|} = \lim\limits_{x\to-\infty} \dfrac{1}{|x|} = 0$.

Example 1.31. Recall the sinc function defined by

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0\,, \\ 1 & \text{if } x = 0\,. \end{cases}$$

Then $\left|\dfrac{\sin x}{x}\right| \leqslant \dfrac{1}{|x|}$ for all $x \neq 0$ and this provides the inequality $-\dfrac{1}{|x|} \leqslant \dfrac{\sin x}{x} \leqslant \dfrac{1}{|x|}$ for all $x \neq 0$. By the Squeeze Theorem and the previous example, we find that

$$\lim_{x\to\infty} \operatorname{sinc}(x) = \lim_{x\to-\infty} \operatorname{sinc}(x) = 0\,.$$

Theorem 1.32

Let f be a function defined on an open interval, and $g(x) = f\left(\dfrac{1}{x}\right)$ if $x \neq 0$.

1. Suppose that the open interval is not bounded from above. Then $\lim\limits_{x\to\infty} f(x)$ exists if and only if $\lim\limits_{x\to 0^+} g(x)$ exists. In either case,
$$\lim_{x\to\infty} f(x) = \lim_{x\to 0^+} g(x)\,.$$
2. Suppose that the open interval is not bounded from below. Then $\lim\limits_{x\to-\infty} f(x)$ exists if and only if $\lim\limits_{y\to 0^-} g(x)$ exists. In either case,
$$\lim_{x\to-\infty} f(x) = \lim_{x\to 0^-} g(x)\,.$$

The theorem above should be very intuitive, and the proof is left as an exercise.

Corollary 1.33

Let p and q be polynomial functions.

1. If the degree of p is smaller than the degree of q, then

$$\lim_{x\to\infty}\frac{p(x)}{q(x)} = \lim_{x\to-\infty}\frac{p(x)}{q(x)} = 0\,.$$

2. If the degree of p is the same as the degree of q, then

$$\lim_{x\to\infty}\frac{p(x)}{q(x)} = \lim_{x\to-\infty}\frac{p(x)}{q(x)} = \frac{\text{the leading coefficient of } p}{\text{the leading coefficient of } q}\,.$$