微積分 MA1001-A 上課筆記（精簡版）

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Definition 1.7

Let f be a function defined on an open interval containing c (except possibly at c), and L be a real number. The statement

$$\lim_{x\to c} f(x) = L\,, \quad \text{read “the limit of } f \text{ at } c \text{ is } L\text{”},$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \;\text{ if }\; 0 < |x - c| < \delta.$$

Theorem 1.12

Let b, c be real numbers, f, g be functions with $\lim\limits_{x\to c} f(x) = L$, $\lim\limits_{x\to c} g(x) = K$. Then

1. $\lim\limits_{x\to c} b = b$, $\lim\limits_{x\to c} x = c$, $\lim\limits_{x\to c} |x| = |c|$;
2. $\lim\limits_{x\to c} \big[f(x) \pm g(x)\big] = L + K$;（和或差的極限等於極限的和或差）
3. $\lim\limits_{x\to c} \big[f(x)g(x)\big] = LK$;（乘積的極限等於極限的乘積）
4. $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$ if $K \neq 0$.（若分母極限不為零，則商的極限等於極限的商）

Theorem 1.15

If $c > 0$ and n is a positive integer, then $\lim\limits_{x\to c} x^{\frac{1}{n}} = c^{\frac{1}{n}}$.

Theorem 1.16

If f and g are functions such that $\lim\limits_{x\to c} g(x) = K$, $\lim\limits_{x\to K} f(x) = L$ and $L = f(K)$, then

$$\lim_{x\to c}(f \circ g)(x) = L\,.$$

Theorem 1.18: Squeeze Theorem（夾擠定理）

Let f, g, h be functions defined on an interval containing c (except possibly at c), and $h(x) \leqslant f(x) \leqslant g(x)$ if $x \neq c$. If $\lim\limits_{x\to c} h(x) = \lim\limits_{x\to c} g(x) = L$, then $\lim\limits_{x\to c} f(x)$ exists and is equal to L.

Example 1.20. In this example we consider the limit of the sine function at a real number

c. Before proceeding, let us first establish a fundamental inequality

$$|\sin x| \leqslant |x| \qquad \text{for all real numbers } x \text{ (in radian unit).} \tag{1.2.1}$$

To see (1.2.1), it suffices to consider the case when $0 < x < \dfrac{\pi}{2}$ for otherwise

1. it trivially holds that $|\sin x| \leqslant x$ if $x = 0$ or $x \geqslant \dfrac{\pi}{2}$;
2. if $x < 0$, then $|\sin x| = |\sin(-x)| \leqslant |-x| = |x|$.

Now suppose that $0 < x < \dfrac{\pi}{2}$. Consider x as a central angle (in radian unit) of a circle of radius 1. Then $\dfrac{\sin x}{2}$ is the largest area of triangles inside the sector, while $\dfrac{x}{2}$ is the area of the sector. Since the area of the sector is larger than the area of triangles inside the sector, we conclude (1.2.1) for the case $0 < x < \dfrac{\pi}{2}$.

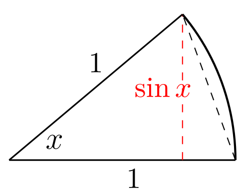

Figure 1.5: The area of the black triangle is smaller than the area of the sector

Now note that the sum and difference formulas

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$$

provide that

$$\begin{aligned}
&\sin x - \sin c \\
&\qquad = \sin\Big(\frac{x+c}{2} + \frac{x-c}{2}\Big) - \sin\Big(\frac{x+c}{2} - \frac{x-c}{2}\Big) \\
&\qquad = \sin\frac{x+c}{2}\cos\frac{x-c}{2} + \sin\frac{x-c}{2}\cos\frac{x+c}{2} - \Big[\sin\frac{x+c}{2}\cos\frac{x-c}{2} - \sin\frac{x-c}{2}\cos\frac{x+c}{2}\Big] \\
&\qquad = 2\sin\frac{x-c}{2}\cos\frac{x+c}{2};
\end{aligned}$$

thus using (1.2.1),

$$|\sin x - \sin c| \leqslant 2\Big|\sin\frac{x-c}{2}\Big| \leqslant |x-c| \qquad \text{for all real number } x.$$

Therefore, $\sin c - |x-c| \leqslant \sin x \leqslant \sin c + |x-c|$ for all real number x, and the Squeeze Theorem then implies that $\lim\limits_{x\to c} \sin x = \sin c$ since $\lim\limits_{x\to c} |x-c| = 0$.

Similarly, using the sum and difference formulas

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi\,,$$

we can also conclude that $\lim\limits_{x\to c}\cos x = \cos c$. The detail is left as an exercise.

By Theorem 1.12, Example 1.20 shows the following

Theorem 1.21

Let c be a real number in the domain of the given trigonometric functions.

1. $\lim\limits_{x\to c}\sin x = \sin c$; 2. $\lim\limits_{x\to c}\cos x = \sin c$; 3. $\lim\limits_{x\to c}\tan x = \tan c$;

4. $\lim\limits_{x\to c}\cot x = \cot c$; 5. $\lim\limits_{x\to c}\sec x = \sec c$; 6. $\lim\limits_{x\to c}\csc x = \csc c$.

Example 1.22. In this example we compute $\lim\limits_{x\to 0} x\sin\dfrac{1}{x}$ if it exists. Note that if the limit exists, we cannot apply 3 of Theorem 1.12 to find the limit since $\lim\limits_{x\to 0}\sin\dfrac{1}{x}$ does not exist. On the other hand, since $\big|x\sin\dfrac{1}{x}\big| \leqslant |x|$ if $x \neq 0$, $-|x| \leqslant x\sin\dfrac{1}{x} \leqslant |x|$ if $x \neq 0$. By the fact that $\lim\limits_{x\to 0}|x| = \lim\limits_{x\to 0}(-|x|) = 0$, the Squeeze Theorem implies that $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$.

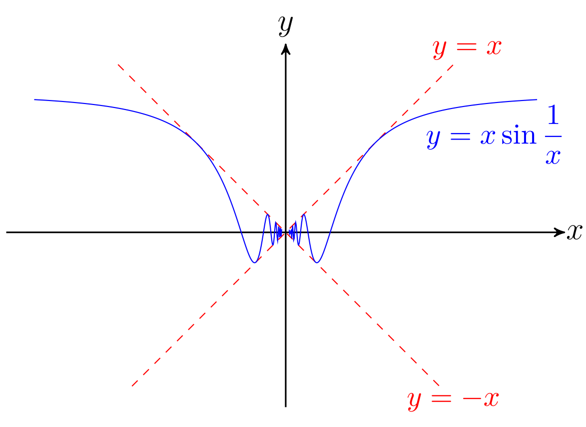

Figure 1.6: The graph of function $y = x\sin\dfrac{1}{x}$

1.2.1 One-sided limits and limits as $x \to \pm\infty$

Suppose that f is a function defined (only) on one side of a point c, it is also possible to consider the one-sided limit $\lim\limits_{x\to c^+} f(x)$ or $\lim\limits_{x\to c^-} f(x)$, where the notation $x \to c^+$ and $x \to c^-$

means that x is taken from the right-hand side and left-hand side of c, respectively, and becomes arbitrarily close to c. In other words, $\lim_{x\to c^+} f(x)$ means the value to which $f(x)$ approaches as x approaches to c from the right, while $\lim_{x\to c^-} f(x)$ means the value to which $f(x)$ approaches as x approaches to c from the left.

Definition 1.23: One-sided limits

Let f be a function defined on an interval with c as the left/right end-point, and L be a real number. The statement

$$\lim_{x\to c^+} f(x) = L \Big/ \lim_{x\to c^-} f(x) = L\,,$$

read "the right/left(-hand) limit of f at c is L" or "the limit of f at c from the right/left is L", means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \;\text{ if }\; 0 < (x - c) < \delta \;/\; -\delta < x - c < 0.$$

Example 1.24. In this example we show that $\lim_{x\to 0^+} x^{\frac{1}{n}} = 0$. Let $\varepsilon > 0$ be given. Define $\delta = \varepsilon^n$. Then $\delta > 0$ and if $0 < x < \delta$, we have

$$|x^{\frac{1}{n}} - 0| = x^{\frac{1}{n}} < \delta^{\frac{1}{n}} = \epsilon\,.$$

We note that Theorem 1.12, Corollary 1.14, Theorem 1.15, 1.16 and 1.18 are also valid when the limits are replaced by one-sided limits (and the precise statements will be provided in the next lecture).

Theorem 1.25

Let f be a function defined on an open interval containing c (except possibly at c). The limit $\lim_{x\to c} f(x)$ exists if and only if $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ both exist and are identical. In either case,

$$\lim_{x\to c} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)\,.$$

Example 1.26. In this example we compute a very important limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\,. \tag{1.2.2}$$

To see this, we first establish the inequality

$$\sin x \leqslant x \leqslant \tan x \qquad \text{for all } 0 < x < \frac{\pi}{2}\,. \tag{1.2.3}$$

We have shown that $\sin x \leqslant x$ if $0 < x < \dfrac{\pi}{2}$ in Example 1.20. For the other part of the inequality, again we consider x as a central angle (in radian unit) of a circle of radius 1. Then $\dfrac{\tan x}{2}$ is the area of the smallest right triangle containing the sector, while $\dfrac{x}{2}$ is the area of the sector. Since the area of the sector is smaller than the area of triangle containing the sector, we conclude that $x \leqslant \tan x$ for the case $0 < x < \dfrac{\pi}{2}$.

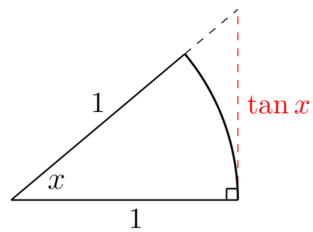

Figure 1.7: The area of the sector is smaller than the area of the triangle

Now using (1.2.3), we find that

$$\cos x \leqslant \frac{\sin x}{x} \leqslant 1 \quad \text{for all } 0 < x < \frac{\pi}{2}\,.$$

The Squeeze Theorem (for one-sided limits) then implies that $\lim\limits_{x\to 0^+} \dfrac{\sin x}{x} = 1$. On the other hand,

$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^-} \frac{\sin(-x)}{-x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1\,;$$

thus Theorem 1.25 implies that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.