微積分 MA1001-A 上課筆記（精簡版）

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Definition 1.7

Let f be a function defined on an open interval containing c (except possibly at c), and L be a real number. The statement

$$\lim_{x\to c} f(x) = L\,, \quad \text{read "the limit of } f \text{ at } c \text{ is } L\text{"},$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \;\text{ if }\; 0 < |x - c| < \delta.$$

Theorem 1.12

Let b, c be real numbers, f, g be functions with $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = K$. Then

1. $\lim_{x\to c} b = b$, $\lim_{x\to c} x = c$, $\lim_{x\to c} |x| = |c|$;
2. $\lim_{x\to c} \big[f(x) \pm g(x)\big] = L + K$;（和或差的極限等於極限的和或差）
3. $\lim_{x\to c} \big[f(x)g(x)\big] = LK$;（乘積的極限等於極限的乘積）
4. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$ if $K \neq 0$.（若分母極限不為零，則商的極限等於極限的商）

Proof. 4. W.L.O.G. (Without loss of generality), we can assume that $K > 0$ for otherwise we have $\lim_{x\to c}(-g)(x) = -K > 0$ and

$$\lim_{x\to c}\Big(\frac{f}{g}\Big)(x) = \lim_{x\to c}\Big(\frac{-f}{-g}\Big)(x) = \frac{\lim_{x\to c}(-f)(x)}{-K} = \frac{-L}{-K} = \frac{L}{K}\,.$$

Let $\varepsilon > 0$ be given. Since $\lim_{x\to c} g(x) = K$, there exist $\delta_1, \delta_2 > 0$ such that

$$|g(x) - K| < \frac{K}{2} \qquad \text{if} \quad 0 < |x - c| < \delta_1$$

and

$$|g(x) - K| < \frac{K^2\varepsilon}{4(|L|+1)} \qquad \text{if} \quad 0 < |x - c| < \delta_2\,.$$

Moreover, since $\lim_{x\to c} f(x) = L$, there exists $\delta_3 > 0$ such that

$$|f(x) - L| < \frac{K\varepsilon}{4} \qquad \text{if} \quad 0 < |x - c| < \delta_3\,.$$

Define $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then $\delta > 0$ and if $0 < |x-c| < \delta$, we have

$$\Big|\frac{f(x)}{g(x)} - \frac{L}{K}\Big| = \frac{|Kf(x) - Lg(x)|}{K|g(x)|} \leqslant \frac{1}{|g(x)|}\frac{|Kf(x) - KL| + |KL - Lg(x)|}{K}$$
$$\leqslant \frac{2}{K}\Big(|f(x) - L| + \frac{|L|}{K}|g(x) - K|\Big)$$
$$< \frac{2}{K}\Big(\frac{K\varepsilon}{4} + \frac{|L|}{K}\frac{K^2\varepsilon}{4(|L|+1)}\Big) \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,$$

where we have used $\dfrac{2}{K} \leqslant \dfrac{1}{|g(x)|}$ if $0 < |x-c| < \delta$ to conclude the inequality. Therefore, we conclude that $\displaystyle\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{K}$ if $K > 0$. □

Theorem 1.15

If $c > 0$ and n is a positive integer, then $\displaystyle\lim_{x\to c} x^{\frac{1}{n}} = c^{\frac{1}{n}}$.

Proof. Let $\varepsilon > 0$ be given. Define $\delta = \min\Big\{\dfrac{c}{2}, \dfrac{nc^{\frac{n-1}{n}}\varepsilon}{2}\Big\}$. Then $\delta > 0$ and if $0 < |x-c| < \delta$, we must have

$$x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}}c^{\frac{1}{n}} + x^{\frac{n-3}{n}}c^{\frac{2}{n}} + \cdots + x^{\frac{1}{n}}c^{\frac{n-2}{n}} + c^{\frac{n-1}{n}} \geqslant \frac{n}{2}c^{\frac{n-1}{n}}\,.$$

Therefore, if $0 < |x-c| < \delta$,

$$\big|x^{\frac{1}{n}} - c^{\frac{1}{n}}\big| = \left|\frac{x-c}{x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}}c^{\frac{1}{n}} + x^{\frac{n-3}{n}}c^{\frac{2}{n}} + \cdots + x^{\frac{1}{n}}c^{\frac{n-2}{n}} + c^{\frac{n-1}{n}}}\right|$$
$$\leqslant \frac{2}{n}c^{-\frac{n-1}{n}}|x-c| < \frac{2}{n}c^{-\frac{n-1}{n}}\delta \leqslant \frac{2}{n}c^{-\frac{n-1}{n}}\frac{nc^{\frac{n-1}{n}}\varepsilon}{2} = \varepsilon$$

which implies that $\displaystyle\lim_{x\to c} x^{\frac{1}{n}} = c^{\frac{1}{n}}$. □

Theorem 1.16

If f and g are functions such that $\displaystyle\lim_{x\to c} g(x) = K$, $\displaystyle\lim_{x\to K} f(x) = L$ and $L = f(K)$, then

$$\lim_{x\to c}(f\circ g)(x) = L\,.$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{x\to L} f(x) = L$, there exists $\delta_1 > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{if} \quad 0 < |x - K| < \delta_1 \,.$$

Since $L = f(K)$, the statement above implies that

$$|f(x) - L| < \varepsilon \quad \text{if} \quad |x - K| < \delta_1 \,.$$

Fix such δ_1. Since $\lim_{x\to c} g(x) = K$, there exists $\delta > 0$ such that

$$|g(x) - K| < \delta_1 \quad \text{if} \quad 0 < |x - c| < \delta \,.$$

Therefore, if $0 < |x - c| < \delta$, $|(f \circ g)(x) - L| = |f(g(x)) - L| < \varepsilon$ which concludes the theorem. □

Remark 1.17. In the theorem above, the condition $L = f(K)$ is important, even though intuitively if $g(x) \to K$ as $x \to c$ and $f(x) \to L$ as $x \to K$ then $(f \circ g)(x)$ should approach L as x approaches c. A counter-example is given by the following two functions: f is the function given in Example 1.2 (from the previous lecture) and g is a constant function with value 2. This example/theorem demonstrates an important fact: intuition could be wrong! That is the reason why mathematicians develop the ε-δ language in order to explain ideas of limits rigorously.

Theorem 1.18: Squeeze Theorem（夾擠定理）

Let f, g, h be functions defined on an interval containing c (except possibly at c), and $h(x) \leqslant f(x) \leqslant g(x)$ if $x \neq c$. If $\lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L$, then $\lim_{x\to c} f(x)$ exists and is equal to L.

Proof. Let $\varepsilon > 0$. Since $\lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L$, there exist $\delta_1, \delta_2 > 0$ such that

$$|h(x) - L| < \varepsilon \quad \text{if} \quad 0 < |x - c| < \delta_1$$

and

$$|g(x) - L| < \varepsilon \quad \text{if} \quad 0 < |x - c| < \delta_2 \,.$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and if $0 < |x - c| < \delta$,

$$L - \varepsilon < h(x) \leqslant f(x) \leqslant g(x) < L + \varepsilon$$

which implies that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$. □

Example 1.19. Find $\lim\limits_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$.

Let $f(x) = \dfrac{\sqrt{x+1}-1}{x}$. If $x \neq 0$,

$$f(x) = \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1} \equiv g(x)\,.$$

To see the limit of g, note that

$$\lim_{x\to 0} \sqrt{x+1} = 1 \qquad \text{(by Theorem 1.16);}$$

thus by Theorem 1.12 $\lim\limits_{x\to 0} g(x) = \dfrac{1}{2}$.