微積分 MA1001-A 上課筆記（精簡版）

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Goal: Given a function f defined "near c", find the value of f at x when x is "arbitrarily close" to c. (給定一函數 f，我們想知道「當除 c 之外的點到 c 的距離愈來愈近時，其函數值是否向某數集中」)

Notation: When there exists such a value, the value is denoted by $\lim\limits_{x\to c} f(x)$.

Example 1.1. Let $g(x) = \dfrac{x^2-1}{x-1}$. Then $\text{Dom}(g) = \mathbb{R}\backslash\{1\}$ and $g(x) = x+1$ if $x \neq 1$. Therefore, the graph of g is given by

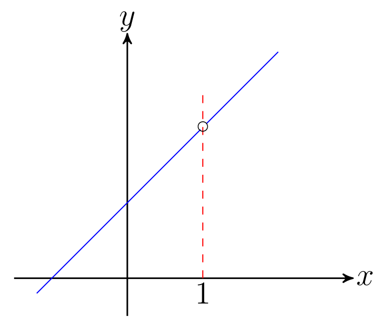

Figure 1.1: The graph of function $g(x) = \dfrac{x^2-1}{x-1}$

Then (by looking at the graph of g we find that) $\lim\limits_{x\to 1} g(x) = 2$.

Example 1.2. Let $f(x) = \begin{cases} 1 & \text{if } x \neq 2\,, \\ 0 & \text{if } x = 2\,. \end{cases}$ The graph of f is given by

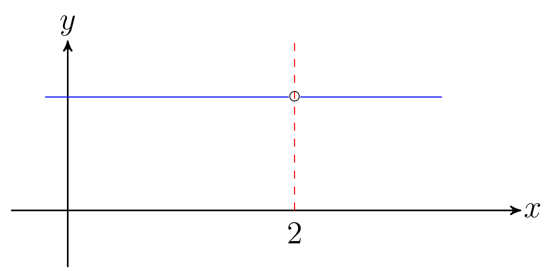

Figure 1.2: The graph of function $f(x)$

Then (by looking at the graph of f we find that) $\lim\limits_{x\to 2} f(x) = 1$.

Next we give some examples in which the limit of functions (at certain points) do not exist.

Example 1.3. (詳見影片) Let $f(x) = \sin\dfrac{1}{x}$. Then $\text{Dom}(f) = \mathbb{R}\backslash\{0\}$. For the graph of f, we note that if $x \in I_n \equiv \left[\dfrac{1}{2n\pi + 2\pi}, \dfrac{1}{2n\pi}\right]$ for some $n \in \mathbb{N}$, the graph of f on I_n must touch

$x = 1$ and $x = -1$ once. Therefore, the graph of f looks like

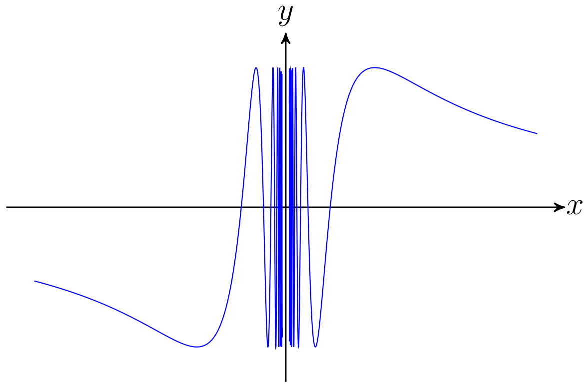

Figure 1.3: The graph of function $f(x) = \sin\frac{1}{x}$

In any interval containing 0, there are infinitely many points whose image under f is 1, and there are always infinitely many points whose image under f is -1. In fact, in any interval containing 0 and $L \in [-1, 1]$ there are infinitely many points whose image under f is L. Therefore, $\lim_{x\to 0} f(x)$ D.N.E. (does not exist).

Example 1.4. Let $f(x) = \frac{|x|}{x}$. Then $f(x) = 1$ if $x > 0$, $f(x) = -1$ if $x < 0$, and the graph of f is given by

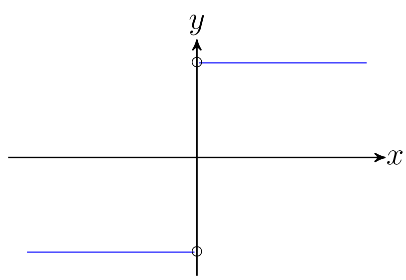

Figure 1.4: The graph of function $f(x) = \frac{|x|}{x}$

By observation (that is, looking at the graph of f), $\lim_{x\to 0} f(x)$ D.N.E.

Example 1.5. （詳見影片）Consider the Dirichlet function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}\,, \\ 1 & \text{if } x \notin \mathbb{Q}\,, \end{cases}$$

where $\mathbb{Q}$ denotes the collection of rational numbers（有理數）. Then $\lim_{x\to c} f(x)$ D.N.E. for all c.

Example 1.6. （詳見影片）Let $f:(0,\infty)\to\mathbb{R}$ be given by

$$f(x)=\begin{cases}\dfrac{1}{p} & \text{if } x=\dfrac{q}{p}, \text{ where } p,q\in\mathbb{N} \text{ and } (p,q)=1\,,\\ 0 & \text{if } x \text{ is irrational（無理數）}.\end{cases}$$

Then $\lim\limits_{x\to c} f(x)=0$ for all $c\in(0,\infty)$.

Definition 1.7

Let f be a function defined on an open interval containing c (except possibly at c), and L be a real number. The statement

$$\lim_{x\to c} f(x)=L\,,\quad \text{read "the limit of } f \text{ at } c \text{ is } L\text{"},$$

means that for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$|f(x)-L|<\varepsilon \ \text{ if } \ 0<|x-c|<\delta.$$

Explanation:（詳見影片）因為 $|f(x)-L|<\varepsilon$ 等價於 $f(x)\in(L-\varepsilon,L+\varepsilon)$，所以定義敘述中的 ε 可視為用來度量 $f(x)$ 向 L 這個數集中的程度。定義所述是指對於任意給定的集中程度 $\varepsilon>0$，一定可以找到在 c 附近的一個範圍（以到 c 的距離小於 δ 來表示），滿足此範圍中的點之函數值落入想要其落入的集中區域 $(L-\varepsilon,L+\varepsilon)$ 之內。此即「當除 c 之外的點到 c 的距離愈來愈近時，其函數值向 L 集中」的意思。

Example 1.8. In this example we show that $\lim\limits_{x\to 1}(x+1)=2$ using Definition 1.7.

Let $\varepsilon>0$ be given. Define $\delta=\varepsilon$. Then $\delta>0$ and if $0<|x-1|<\delta$, we have

$$|(x+1)-2|=|x-1|<\delta=\varepsilon\,.$$

One could also pick $\delta=\dfrac{\varepsilon}{2}$ so that if $0<|x-1|<\delta$,

$$|(x+1)-2|=|x-1|<\delta=\frac{\varepsilon}{2}<\varepsilon\,.$$

Example 1.9. Show that $\lim\limits_{x\to 2}x^2=4$. If $\varepsilon=1$, we can choose $\delta=\min\left\{\sqrt{5}-2,2-\sqrt{3}\right\}$ so that $\delta>0$ and if $0<|x-2|<\delta$ we must have $|x^2-4|<1$.

For general ε, we can choose $\delta=\min\left\{\sqrt{4+\varepsilon}-2,2-\sqrt{4-\varepsilon}\right\}$ so that $\delta>0$ and if $0<|x-2|<\delta$ we must have $|x^2-4|<\varepsilon$.

Proposition 1.10

Let f, g be functions defined on an open interval containing c (except possibly at c), and $f(x) = g(x)$ if $x \neq c$. If $\lim\limits_{x\to c} g(x) = L$, then $\lim\limits_{x\to c} f(x) = L$.

Proof. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to c} g(x) = L$, there exists $\delta > 0$ such that

$$|g(x) - L| < \varepsilon \quad \text{if} \quad 0 < |x - c| < \delta.$$

Since $f(x) = g(x)$ if $x \neq c$, we must have if $0 < |x - c| < \delta$,

$$|f(x) - L| = |g(x) - L| < \varepsilon\,.$$

□

Example 1.11. Let $f(x) = x + 1$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$. Since $f(x) = g(x)$ if $x \neq 1$, the proposition above implies that

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} f(x) = 2\,.$$

1.2 Properties of Limits

Theorem 1.12

Let b, c be real numbers, f, g be functions with $\lim\limits_{x\to c} f(x) = L$ and $\lim\limits_{x\to c} g(x) = K$.

1. $\lim\limits_{x\to c} b = b$, $\lim\limits_{x\to c} x = c$, $\lim\limits_{x\to c} |x| = |c|$;
2. $\lim\limits_{x\to c} \big[f(x) \pm g(x)\big] = L + K$;（和或差的極限等於極限的和或差）
3. $\lim\limits_{x\to c} \big[f(x)g(x)\big] = LK$;（乘積的極限等於極限的乘積）
4. $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$ if $K \neq 0$.（若分母極限不為零，則商的極限等於極限的商）

Example 1.13. Find $\lim\limits_{x\to 3} x^2$. By 1 of Theorem 1.12 $\lim\limits_{x\to 3} x = 3$; thus 3 of Theorem 1.12 implies that

$$\lim_{x\to 3} x^2 = \Big(\lim_{x\to 3} x\Big)\Big(\lim_{x\to 3} x\Big) = 9\,.$$

The above equality further shows that

$$\lim_{x\to 3} x^3 = \Big(\lim_{x\to 3} x^2\Big)\Big(\lim_{x\to 3} x\Big) = 27\,.$$

In particular, if n is a positive integer, then (by induction) $\lim_{x \to c} x^n = c^n$.

Corollary 1.14

Assume the assumptions in Theorem 1.12, and let n be a positive integer.

1. $\lim_{x \to c} \left[f(x)^n \right] = L^n$.
2. If p is a polynomial function, then $\lim_{x \to c} p(x) = p(c)$.
3. If r is a rational function given by $r(x) = \dfrac{p(x)}{q(x)}$ for some polynomials p and q, and $q(c) \neq 0$, then $\lim_{x \to c} r(x) = r(c)$.

An illustration of why 2 in Corollary 1.13 is correct: Suppose that $p(x) = 3x^2 + 5x - 10$. Then applying 1-3 in Theorem 1.12, we obtain that

$$\begin{aligned}
\lim_{x \to c} p(x) &= \lim_{x \to c}(3x^2 + 5x) - \lim_{x \to c}(10) = \lim_{x \to c}(3x^2 + 5x) - 10 \\
&= \Big(\lim_{x \to c}(3) \Big) \Big(\lim_{x \to c} x^2 \Big) + \Big(\lim_{x \to c}(5) \Big) \Big(\lim_{x \to c} x \Big) - 10 \\
&= 3c^2 + 5c - 10 = p(c)\,.
\end{aligned}$$