微積分 MA1001-A 上課筆記（精簡版）

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Ching-hsiao Arthur Cheng 鄭經斅

Chapter 0

Preliminary

0.1 Functions and Their Graphs

Definition 0.1: Real-Valued Functions of a Real Variable

Let $X, Y \subseteq \mathbb{R}$ be subsets of real numbers. A real-valued function f of a real variable x from X to Y is a correspondence that assigns to each element x in X exactly one number y in Y. Here X is called the domain of f and is usually denoted by $\mathrm{Dom}(f)$, Y is called "the" co-domain of f, the number y is called the image of x under f and is usually denoted by $f(x)$, which is called the value of f at x. The range of f, denoted by $\mathrm{Ran}(f)$, is a subset of Y consisting of all images of numbers in X. In other words,

$$\mathrm{Ran}(f) \equiv \text{the range of } f \equiv \left\{ f(x) \,\middle|\, x \in X \right\}.$$

Remark 0.2. Given a way of assignment $x \mapsto f(x)$ without specifying where x is chosen from, we still treat f as a function and $\mathrm{Dom}(f)$ is considered as the collection of all $x \in \mathbb{R}$ such that $f(x)$ is well-defined. For example, $f(x) = x + 1$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$ are both considered as functions with

$$\mathrm{Dom}(f) = \mathbb{R} \qquad \text{and} \qquad \mathrm{Dom}(g) = \mathbb{R} \backslash \{1\}.$$

Since $\mathrm{Dom}(f) \neq \mathrm{Dom}(g)$, f and g are considered as different functions even though $f(x) = g(x)$ for all $x \neq 1$.

Terminologies:

1. Explicit form of a function: $y = f(x)$;

2. Implicit form of a function: $F(x, y) = 0$. (参考影片)

Definition 0.3

A function f is a polynomial function if f takes the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\,,$$

where $a_0, a_1, a_2, \cdots, a_n$ are real numbers, called coefficients of the polynomial, and n is a non-negative integer. If $a_n \neq 0$, then a_n is called the leading coefficient, and n is called the degree of the polynomial. A rational function is the quotient of two polynomials.

Definition 0.4

The graph of the function $y = f(x)$ consists of all points $\big(x, f(x)\big)$, where x is in the domain of f. In other words,

$$\mathrm{G}(f) \equiv \text{the graph of } f \equiv \Big\{\big(x, f(x)\big)\,\Big|\, x \in \mathrm{Dom}(f)\Big\}\,.$$

Definition 0.5: Composite Functions

Let f and g be functions. The function $f \circ g$, read f circle g, is the function defined by $(f \circ g)(x) = f\big(g(x)\big)$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words,

$$\mathrm{Dom}(f \circ g) = \big\{x \in \mathrm{Dom}(g)\,\big|\, g(x) \in \mathrm{Dom}(f)\big\}\,.$$

Chapter 1

Limits and Continuity

1.1 Limits of Functions

Goal: Given a function f defined "near c", find the value of f at x when x is "arbitrarily close" to c.

Example 1.1. Consider the function $g(x) = \dfrac{x^2 - 1}{x - 1}$ given in Remark 0.2, and

$$h(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{if } x \neq 1\,, \\ 0 & \text{if } x = 1\,. \end{cases}$$

Then the limit of g at 1 should be the same as the limit of h at 1. Therefore, to consider the limit of a function at a point c, the value of the function at c is not important at all.