微積分 MA1001-A 上課筆記（精簡版）

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7.6 Moments, Centers of Mass, and Centroids

• Center of mass in a one-dimensional system

Let $m_1, m_2, \cdots, m_n$ be n point masses located at $x_1, x_2, \cdots, x_n$ on a (massless) rigid x-axis supported by a fulcrum at the origin.

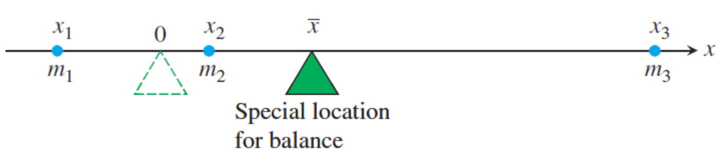
Each mass m_k exerts a downward force $m_k g$ (which is negative), and each of these forces has a tendency to turn the x-axis about the origin. This turning effect, called a torque, is measured by multiplying the force $m_k g$ by the signed distance x_k from the point of application to the origin. By convention, a positive torque induces a counterclockwise turn.

The sum of these torques measures the tendency of the system to rotate about the fultrum/origin. This sum is called the system torque; thus

$$\text{System torque} = m_1 g x_1 + m_2 g x_2 + \cdots + m_n g x_n = g(m_1 x_1 + m_2 x_2 + \cdots + m_n x_n)\,.$$

The system will balance if and only if its torque is zero. The number $M_0 \equiv m_1 x_1 + m_2 x_2 + \cdots + m_n x_n$ is called the moment of the system about the origin, and is the sum of moments $m_1 x_1$, $m_2 x_2$, $\cdots$, $m_n x_n$ of individual masses. If M_0 is 0, then the system is said to be in equilibrium.

For a system that is not in equilibrium, the center of mass (of the system) is defined as the point $\bar{x}$ at which the fulcrum could be relocated to attain equilibrium.



Such an $\bar{x}$ must satisfy

$$0 = m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) + \cdots + m_n(x_n - \bar{x})$$

which implies that

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{\text{moment of system about the origin}}{\text{total mass of system}}\,.$$

Definition 7.20

Let the point masses $m_1, m_2, \cdots, m_n$ be located at x_1, x_2, $\cdots$, x_n (on a coordinate line).

1. The moment about the origin is
$$M_0 = m_1x_1 + m_2x_2 + \cdots + m_nx_n\,.$$
2. The center of mass $\bar{x}$ is $\dfrac{M_0}{m}$, where $m = m_1 + m_2 + \cdots + m_n$ is the total mass of the system.

- **Center of mass in a two-dimensional system**

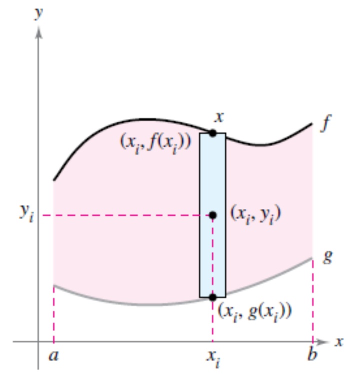
Definition 7.21

Let the point masses $m_1, m_2, \cdots, m_n$ be located at (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n) (on a plane).

1. The moment about the y-axis is
$$M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n\,.$$
2. The moment about the x-axis is
$$M_x = m_1y_1 + m_2y_2 + \cdots + m_ny_n\,.$$
3. The center of mass $(\bar{x}, \bar{y})$ is
$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}\,,$$
where $m = m_1 + m_2 + \cdots + m_n$ is the total mass of the system.

- **Center of mass of a planar lamina**

Consider an irregularly shaped thin flat plate of material (called lamina) of uniform density ϱ (a measure of mass per unit of area), bounded by the graphs of $y = f(x)$, $y = g(x)$, and $x = a$, $x = b$, as shown in the following figure.



Then the density of this region is

$$m = \varrho \int_a^b \big[f(x) - g(x)\big]\,dx = \varrho A\,,$$

where A is the area of this region.

Partition $[a, b]$ into n sub-intervals with equal width Δx, and let x_i be the mid-point of the i-th sub-interval. The area of the portion on the i-th sub-interval can be approximated by $\big[f(x_i) - g(x_i)\big]\Delta x$; thus the mass of the portion on the i-th sub-interval can be approximated by $\varrho\big[f(x_i)-g(x_i)\big]\Delta x$. Now, considering this mass to be located at the center $\big(x_i, \dfrac{f(x_i)+g(x_i)}{2}\big)$, the moment of this mass about the x-axis is

$$\varrho\big[f(x_i) - g(x_i)\big]\Delta x \frac{f(x_i)+g(x_i)}{2}\,.$$

Summing all the moments and passing to the limit as $n \to \infty$ suggest the following

Definition 7.22

Let $f, g : [a, b] \to \mathbb{R}$ be continuous such that $f(x) \geqslant g(x)$ for all $x \in [a, b]$, and consider the lamina of uniform density ϱ bounded by the graphs of f, g and the lines $x = a$, $x = b$.

1. The moment about the x-axis and the y-axis are

$$M_x = \frac{\varrho}{2}\int_a^b \big[f(x)^2 - g(x)^2\big]\,dx \quad \text{and} \quad M_y = \varrho\int_a^b x\big[f(x) - g(x)\big]\,dx\,.$$

2. The center of mass $(\bar{x}, \bar{y})$ is given by $\bar{x} = \dfrac{M_y}{m}$ and $\bar{y} = \dfrac{M_x}{m}$, where $m = \varrho\displaystyle\int_a^b \big[f(x) - g(x)\big]\,dx$ is the mass of the lamina.

The center of mass of a lamina of uniform density depends only on the shape of the lamina but not on its density. For this reason, the center of mass of a region in the plain is also called the centroid of the region.

Example 7.22. Compute the centroid of a triangle with vertex $(0,0)$, (a,b_1) and (a,b_2), where $a>0$ and $b_1<b_2$.

Let $f(x)=\frac{b_2}{a}x$ and $g(x)=\frac{b_1}{a}x$. Then the triangle given above is the region bounded by the graphs of f, g and $x=a$. Assume uniform density $\varrho=1$. Then the moment of the region about the x-axis is

$$M_x=\frac{1}{2}\int_0^a \Big(\frac{b_2^2}{a^2}-\frac{b_1^2}{a^2}\Big)x^2\,dx=\frac{a(b_2^2-b_1^2)}{6}$$

and the moment of the region about the y-axis is

$$M_y=\int_0^a x\Big[\frac{b_2}{a}-\frac{b_1}{a}\Big]x\,dx=\frac{a^2(b_2-b_1)}{3}\,,$$

as well as the total mass

$$m=\int_0^a \Big[\frac{b_2}{a}-\frac{b_1}{a}\Big]x\,dx=\frac{a(b_2-b_1)}{2}\,.$$

Therefore, the centroid of the given triangle is

$$(\bar{x},\bar{y})=\Big(\frac{2a}{3},\frac{b_1+b_2}{3}\Big)\,.$$

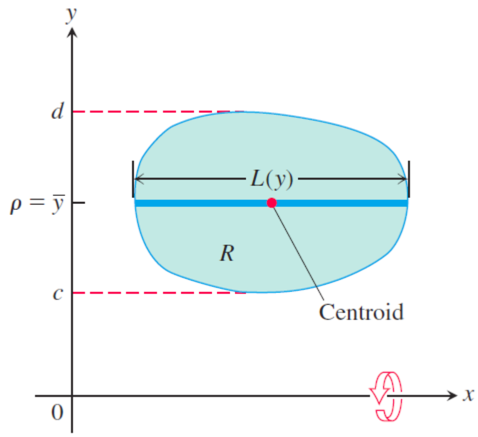
Theorem 7.23: Pappus

Let R be a region in a plane and L be a line in the same plane such that L does not intersect the interior of R. If r is the distance between the centroid of R and the line, then the volume V of the solid of revolution formed by revolving R about the line is

$$V=2\pi rA\,,$$

where A is the area of R.

Proof. We draw the axis of revolution as the x-axis with the region R in the first quadrant (see figure below).



Let $L(y)$ be the length of the cross section of R perpendicular to the y-axis at y, and we assume that L is continuous on $[c, d]$. Then the area of R is given by

$$A = \int_c^d L(y)\, dy\,,$$

and the shell method implies that the volume of the solid formed by revolving R about the x-axis is

$$V = 2\pi \int_c^d yL(y)\, dy\,.$$

On the other hand, if r denotes the distance between the centroid of R and the x-axis, then r is the y-coordinate of the centroid of R and is given by

$$r = \frac{\text{the moment of the region about the } x\text{-axis}}{\text{the total mass of the region}} = \frac{\displaystyle\int_c^d yL(y)\, dy}{\displaystyle\int_c^d L(y)\, dy}$$

which validates the relation $V = 2\pi rA$. □

Example 7.24. Using the Pappus theorem, the volume of the solid torus given in Example 7.2 is

$$2\pi a(\pi r^2) = 2\pi^2 ar^2$$

since the centroid of a disk is the center of the disk.