微積分 MA1001-A 上課筆記（精簡版）

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7.4 Arc Length and Surfaces of Revolution

7.4.1 Arc length

In this sub-section we consider the arc length of a curve which is the graph of a function on a closed interval. Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$. Then the arc length of the graph of f on $[a, b]$ can be approximated by

$$\sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + \big(f(x_k) - f(x_{k-1})\big)^2}\,,$$

where for each k, the number $\sqrt{(x_k - x_{k-1})^2 + \big(f(x_k) - f(x_{k-1})\big)^2}$ is the length of the line segment joining points $\big(x_k, f(x_k)\big)$ and $\big(x_{k-1}, f(x_{k-1})\big)$.

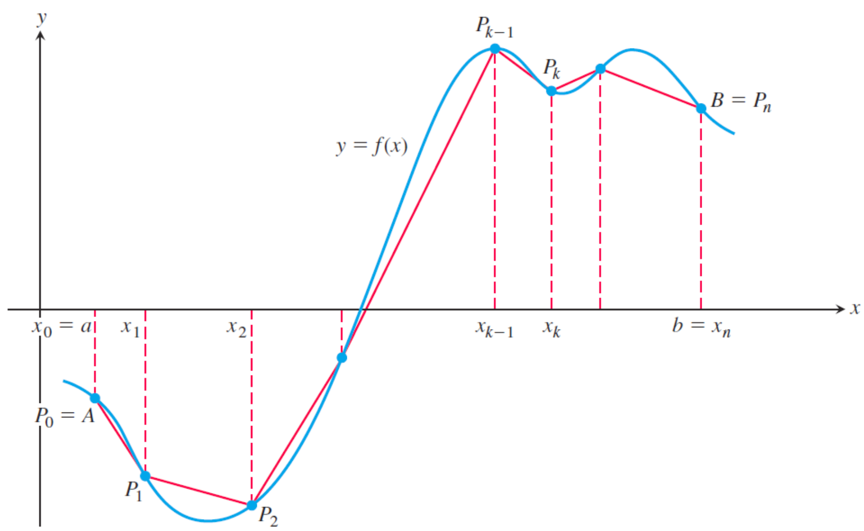

Figure 7.1: The length of the polygonal path $P_0P_1P_2 \cdots P_n$ approximates the arc length of the graph of f on $[a, b]$

With Δx_k denoting $x_k - x_{k-1}$, then

$$\sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + \big(f(x_k) - f(x_{k-1})\big)^2} = \sum_{k=1}^{n} \sqrt{1 + \Big(\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\Big)^2}\,\Delta x_k\,.$$

If f is differentiable on (a, b), then the Mean Value Theorem implies that for each $1 \leqslant k \leqslant n$ there exists $c_k \in (x_{k-1}, x_k)$ such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(c_k)\,;$$

thus

$$\sum_{k=1}^{n} \sqrt{(x_k - x_{k-1})^2 + \big(f(x_k) - f(x_{k-1})\big)^2} = \sum_{k=1}^{n} \sqrt{1 + f'(c_k)^2}\Delta x_k$$

which is a Riemann sum of the function $y = \sqrt{1 + f'(x)^2}$ for partition $\mathcal{P}$. Therefore, if f is continuously differentiable on $[a, b]$; that is, f' is continuous on $[a, b]$, then $\sqrt{1 + f'(x)^2}$ is Riemann integrable on $[a, b]$. Therefore, using the arguments in Section **??**, we find that the arc length of the graph of a continuously differentiable function f on $[a, b]$ is

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx\,.$$

Example 7.1. Compute the perimeter of a circle with radius r.

Let $f(x) = \sqrt{r^2 - x^2}$. Then the perimeter of a circle with radius r is the same as twice the arc length of the graph of f on $[-r, r]$. Therefore, the perimeter of a circle with radius r is

$$\begin{aligned} 2\int_{-r}^{r} \sqrt{1 + f'(x)^2}\, dx &= 2\int_{-r}^{r} \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = 2r\int_{-r}^{r} \frac{1}{\sqrt{r^2 - x^2}}\, dx \\ &= 2r\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\sqrt{r^2 - r^2\sin^2 u}} r\cos u\, du \\ &= 2r\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r\cos u}{r\cos u}\, du = 2r\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} du = 2\pi r\,. \end{aligned}$$

Example 7.2. The arc length of the graph of $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on the interval $\big[\frac{1}{2}, 2\big]$ is

$$\begin{aligned} \int_{\frac{1}{2}}^{2} \sqrt{1 + \Big[\frac{d}{dx}\big(\frac{x^3}{6} + \frac{1}{2x}\big)\Big]^2}\, dx &= \int_{\frac{1}{2}}^{2} \sqrt{1 + \Big[\frac{x^2}{2} - \frac{1}{2x^2}\Big]^2}\, dx \\ = \int_{\frac{1}{2}}^{2} \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}}\, dx &= \int_{\frac{1}{2}}^{2} \sqrt{\Big(\frac{x^2}{2} + \frac{1}{2x^2}\Big)^2}\, dx \\ = \int_{\frac{1}{2}}^{2} \Big(\frac{x^2}{2} + \frac{1}{2x^2}\Big)\, dx &= \Big(\frac{x^3}{6} - \frac{1}{2x}\Big)\Big|_{x=\frac{1}{2}}^{x=2} = \frac{33}{16}\,. \end{aligned}$$

Example 7.3. The arc length of the graph of $y = \ln(\cos x)$ from $x = 0$ to $x = \dfrac{\pi}{4}$ is

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sqrt{1 + \big(\frac{d}{dx}\ln(\cos x)\big)^2}\, dx &= \int_0^{\frac{\pi}{4}} \sqrt{1 + \frac{\sin^2 x}{\cos^2 x}}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x}\, dx \\ &= \int_0^{\frac{\pi}{4}} \sec x\, dx = \ln|\sec x + \tan x|\Big|_{x=0}^{x=\frac{\pi}{4}} \\ &= \ln(\sqrt{2} + 1) - \ln 1 = \ln(\sqrt{2} + 1)\,. \end{aligned}$$

Let f be continuously differentiable on $[a, b]$. Then the arc length of the graph of f on $[a, x]$, where $x \in [a, b]$, is given by

$$s(x) = \int_a^x \sqrt{1 + f'(t)^2}\, dx\,.$$

The fundamental theorem of Calculus then shows that

$$s'(x) = \frac{ds}{dx}(x) = \sqrt{1 + f'(x)^2}$$

or equivalently,

$$\frac{ds}{dx} = \sqrt{1 + \Big(\frac{dy}{dx}\Big)^2}\,.$$

Symbolically, $ds = \sqrt{dx^2 + dy^2}$; thus the arc length of the graph of a function is $\displaystyle\int ds$. This variable s is usually called the arc length parameter.

7.4.2 Surface of Revolution

In this section we consider the surface area of a surface formed by revolving a curve about a line (again, this line is called the axis of revolution and usually is a line parallel to the x-axis or y-axis). Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and $\mathcal{S}$ be the surface formed by revolving the graph of f on $[a, b]$ about the x-axis. The general procedures shown in the previous sections is first finding a way to compute an approximated value of the surface area and then see what is the limit of this approximation as $\|\mathcal{P}\|$ approaches 0.

We first try the following idea: let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$ and $\Delta x_k = x_k - x_{k-1}$. Consider the sum

$$\sum_{k=1}^{n} 2\pi f(c_k)\Delta x_k\,, \qquad c_k \in [x_{k-1}, x_k]$$

which is the sum of the area of cylinders formed by revolving the graph of the constant function $y = f(c_k)$ on $[x_{k-1}, x_k]$ about the x-axis. Since the sum above is a Riemann sum of the function $y = 2\pi f(x)$ for partition $\mathcal{P}$, we expect that if $f : [a, b] \to \mathbb{R}$ is continuous, then as $\|\mathcal{P}\| \to 0$ the sum approaches

$$2\pi \int_a^b f(x)\, dx\,.$$

If this is true, then the surface of the sphere with radius r is given by

$$2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cos^2 u \, du = 2\pi r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos(2u)}{2}\, du = \pi^2 r^2$$

which is definitely not the correct area of the sphere with radius r. What is wrong with this idea?

The mistake is due to that the area of surface of revolution has to be approximated by the sum of the lateral surface area of frustum of right circular cones rather than sum of lateral surface area of cylinders. The lateral area of the frustum in Figure 7.2 below

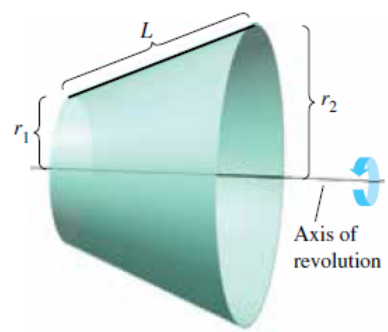

Figure 7.2

is given by $2\pi r L$, where $r = \dfrac{r_1 + r_2}{2}$; thus the surface area of $\mathcal{S}$ can be approximated by

$$\begin{aligned}
&\sum_{k=1}^{n} 2\pi \frac{f(x_k) + f(x_{k-1})}{2} \sqrt{(x_k - x_{k-1})^2 + \big(f(x_k) - f(x_{k-1})\big)^2} \\
&\quad = \sum_{k=1}^{n} 2\pi \frac{f(x_k) + f(x_{k-1})}{2} \sqrt{1 + f'(c_k)^2} \Delta x_k
\end{aligned}$$

It can be shown that the sum above approaches $\displaystyle\int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\, dx$ as $\|\mathcal{P}\|$ approaches 0. Therefore, the area of the surface formed by revolving the graph of f on $[a, b]$ about the x-axis is given by

$$2\pi \int_a^b \big|f(x)\big| \sqrt{1 + f'(x)^2}\, dx \,.$$

In general, the area of the surface formed by revolving the graph of f on $[a, b]$ about $y = L$ is given by

$$2\pi \int_a^b \big|f(x) - L\big| \sqrt{1 + f'(x)^2}\, dx \,.$$

Example 7.4. The surface area of a sphere with radius r is given by

$$2\pi \int_{-r}^{r} \big(\sqrt{r^2 - x^2} - 0\big)\sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx = 2\pi \int_{-r}^{r} \sqrt{r^2}\, dx = 4\pi r^2\,,$$

where we treat the sphere as a surface formed by revolving the graph of $y = \sqrt{r^2 - x^2}$ about the x-axis.

Example 7.5. In this example we consider the area of the surface formed by revolving the (upper part) ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (or the graph of $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$ on $[-a, a]\big)$ about the x-axis. Using the formula of computing the area of surfaces of revolution above, we find that the surface area is given by

$$\begin{aligned}
2\pi \int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\sqrt{1 + \frac{b^2}{a^2}\frac{x^2}{a^2 - x^2}}\, dx &= \frac{2\pi b}{a} \int_{-a}^{a} \sqrt{a^2 - x^2 + \frac{b^2}{a^2}x^2}\, dx \\
&= \frac{2\pi b}{a^2} \int_{-a}^{a} \sqrt{a^4 - (a^2 - b^2)x^2}\, dx = 2\pi b \int_{-a}^{a} \sqrt{1 - \frac{a^2 - b^2}{a^4}x^2}\, dx\,.
\end{aligned}$$

1. Suppose that $a > b$; that is, x-axis is the major axis. Let $c = \dfrac{\sqrt{a^2 - b^2}}{a^2}$. Then

$$2\pi \int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\sqrt{1 + \frac{b^2}{a^2}\frac{x^2}{a^2 - x^2}}\, dx = 2\pi b \int_{-a}^{a} \sqrt{1 - c^2x^2}\, dx\,.$$

   Making use of the substitution $x = \dfrac{1}{c}\sin u$, we find that

$$\begin{aligned}
2\pi \int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\sqrt{1 + \frac{b^2}{a^2}\frac{x^2}{a^2 - x^2}}\, dx &= 2\pi b \int_{-\arcsin(ac)}^{\arcsin(ac)} \sqrt{1 - \sin^2 u} \cdot \frac{1}{c}\cos u\, du \\
&= \frac{2\pi b}{c} \int_{-\arcsin(ac)}^{\arcsin(ac)} \cos^2 u\, du = \frac{2\pi b}{c} \int_{-\arcsin(ac)}^{\arcsin(ac)} \frac{1 + \cos(2u)}{2}\, du \\
&= \frac{2\pi b}{c}\Big(\frac{u}{2} + \frac{\sin(2u)}{4}\Big)\Big|_{u=-\arcsin(ac)}^{u=\arcsin(ac)} \\
&= \frac{2\pi a^2 b}{\sqrt{a^2 - b^2}}\Big[\arcsin\frac{\sqrt{a^2 - b^2}}{a} + \frac{b\sqrt{a^2 - b^2}}{a^2}\Big] \\
&= \frac{2\pi a^2 b}{\sqrt{a^2 - b^2}}\arcsin\frac{\sqrt{a^2 - b^2}}{a} + 2\pi b^2\,.
\end{aligned}$$

2. Suppose that $a < b$; that is, x-axis is the minor axis. Let $c = \dfrac{\sqrt{b^2 - a^2}}{a^2}$. Then

$$2\pi \int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}\sqrt{1 + \frac{b^2}{a^2}\frac{x^2}{a^2 - x^2}}\, dx = 2\pi b \int_{-a}^{a} \sqrt{1 + c^2x^2}\, dx\,.$$

Similar to the previous case, the substitution $x = \dfrac{1}{c}\sinh u$ implies that

$$
\begin{aligned}
2\pi \int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2}\sqrt{1+\frac{b^2}{a^2}\frac{x^2}{a^2-x^2}}\,dx &== 2\pi b \int_{-\sinh^{-1}(ac)}^{\sinh^{-1}(ac)} \sqrt{1+\sinh^2 u}\cdot\frac{1}{c}\cosh u\,du \\
&= \frac{2\pi b}{c}\int_{-\sinh^{-1}(ac)}^{\sinh^{-1}(ac)} \cosh^2 u\,du = \frac{2\pi b}{c}\int_{-\sinh(ac)}^{\sinh^{-1}(ac)} \frac{1+\cosh(2u)}{2}\,du \\
&= \frac{2\pi b}{c}\Big(\frac{u}{2}+\frac{\sinh(2u)}{4}\Big)\Big|_{u=-\sinh^{-1}(ac)}^{u=\sinh^{-1}(ac)} \\
&= \frac{2\pi a^2 b}{\sqrt{b^2-a^2}}\Big[\sinh^{-1}\frac{\sqrt{a^2-b^2}}{a}+\frac{\sqrt{a^2-b^2}}{a}\cosh\Big(\sinh^{-1}\frac{\sqrt{b^2-a^2}}{a}\Big)\Big] \\
&= \frac{2\pi a^2 b}{\sqrt{b^2-a^2}}\sinh^{-1}\frac{\sqrt{a^2-b^2}}{a}+2\pi b^2\,.
\end{aligned}
$$