微積分 MA1001-A 上課筆記（精簡版）

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7.3 Volume: The Shell Method（剝殼法）

Consider the volume of a solid D formed by revolving a region R about line $x = L$, where R is given by

$$\mathrm{R} = \left\{(x, y) \,\middle|\, x \in [a, b]\,, 0 \leqslant y \leqslant f(x)\right\},$$

for some $a \geqslant L$ and continuous function $f : [a, b] \to [0, \infty)$. When f is one-to-one, let $g : [c, d] \to \mathbb{R}$ be the inverse function of f $\big($note that $c = \min\limits_{x\in[a,b]} f(x)$ and $d = \max\limits_{x\in[a,b]} f(x)\big)$. Then the volume of D computed using the disk method is given by

$$\pi \int_c^d \left[(g(y) - L)^2 - (a - L)^2\right]^2 dy + \pi \int_0^c \left[(b - L)^2 - (a - L)^2\right] dy\,.$$

On the other hand, if f is not one-to-one, then it will be not so easy to find the volume of D using the disk method. How do we compute the volume of D in this case?

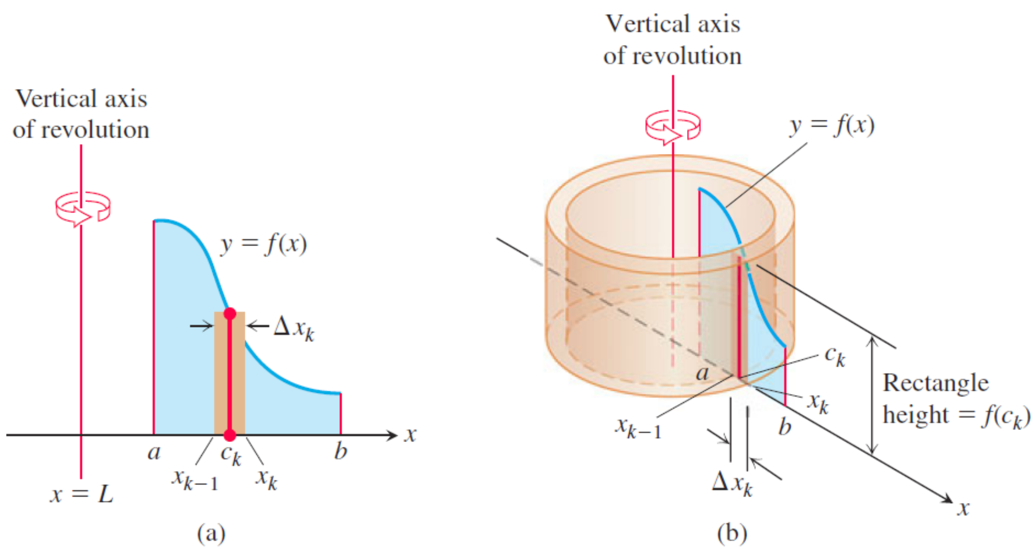


Figure 7.1: The shell method

Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$, $\Delta x_k = x_k - x_{k-1}$ and $c_k = \dfrac{x_{k-1} + x_k}{2}$; that is, c_k is the middle point of the interval $[x_{k-1}, x_k]$. Then Figure 7.1 implies that the volume of D can be approximated by the sum of these cylindrical shells (one cylindrical shell is shown above in orange color). The volume of the cylindrical shell shown in Figure 7.1 is given by

$$\begin{aligned}\pi(x_k-L)^2 f(c_k)-\pi(x_{k-1}-L)^2 f(c_k) &= \pi f(c_k)\big[(x_k-L)^2-(x_{k-1}-L)^2\big]\\ &= \pi f(c_k)(x_k-L+x_{k-1}-L)\big[x_k-L-(x_{k-1}-L)\big] = 2\pi(c_k-L)f(c_k)\Delta x_k\,.\end{aligned}$$

Therefore, the volume of D can be approximated by the sum

$$\sum_{k=1}^{n} 2\pi(c_k-L)f(c_k)\Delta x_k\,.$$

We note that the sum above is a Riemann sum of the function $y = 2\pi(x-L)f(x)$ for partition $\mathcal{P}$ using the mid-point rule. Therefore, similar to the argument in Section **??**, we find that the volume of D is given by

$$2\pi\int_a^b (x-L)f(x)\,dx\,.$$

This way of computing the volume of D is called the shell method.

Similarly, let D be formed by revolving a region R about the line $x = L$, where the region R is given by

$$\mathrm{R} = \big\{(x,y)\,\big|\,x\in[a,b]\,,g(x)\leqslant y\leqslant f(x)\big\}$$

for some $a > L$ and continuous functions $f, g : [a,b]\to\mathbb{R}$ with $\min\limits_{x\in[a,b]} f(x)\geqslant \max\limits_{x\in[a,b]} g(x)$. Then the volume of D is given by

$$2\pi\int_a^b (x-L)\big[f(x)-g(x)\big]\,dx\,.$$

Example 7.1. In this example we compute the volume of the ball $B(0,r)$ by the shell method. Note that $B(0,r)$ can be formed by revolving the region

$$\mathrm{R} = \big\{(x,y)\,\big|\,x\in[0,r]\,,-\sqrt{r^2-x^2}\leqslant y\leqslant \sqrt{r^2-x^2}\big\}$$

about the y-axis. Therefore, by the shell method, the volume of $B(0,r)$ is given by

$$\begin{aligned}&\pi\int_0^r (x-0)\big[\sqrt{r^2-x^2}-(-\sqrt{r^2-x^2})\big]\,dx\\ &\quad = 4\pi\int_0^r x(r^2-x^2)^{\frac{1}{2}}\,dx = 4\pi\Big[-\frac{3}{4}(r^2-x^2)^{\frac{2}{3}}\Big]\Big|_{x=0}^{x=r} = \frac{4\pi}{3}r^3\,.\end{aligned}$$

Example 7.2. In this example we compute the volume of the solid torus given in Example **??** by the shell method. Note that this solid torus can also be formed by revolving the region

$$\mathrm{R} = \big\{(x,y)\,\big|\, x\in[a-r,a+r]\,, -\sqrt{r^2-(x-a)^2} \leqslant y \leqslant \sqrt{r^2-(x-a)^2}\big\}$$

about the y-axis. Therefore, by the shell method, the volume of the solid torus given in Example **??** is given by

$$2\pi\int_{a-r}^{a+r}(x-0)\big[\sqrt{r^2-(x-a)^2}-(-\sqrt{r^2-(x-a)^2})\big]\,dx = 4\pi\int_{a-r}^{a+r}x\sqrt{r^2-(x-a)^2}\,dx\,.$$

By the substitution $x = a + r\sin u$, we find that

$$\begin{aligned}\int_{a-r}^{a+r}x\sqrt{r^2-(x-a)^2}\,dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(a+r\sin u)\sqrt{r^2-r^2\sin^2 u}\cdot r\cos u\,du\\ &= r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(a+r\sin u)\cos^2 u\,du = r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big[a\frac{1+\cos(2u)}{2}+r\sin u\cos^2 u\Big]\,du\\ &= r^2\Big[\frac{au}{2}+\frac{a\sin(2u)}{4}-\frac{r}{3}\cos^3 u\Big]\Big|_{u=-\frac{\pi}{2}}^{u=\frac{\pi}{2}} = \frac{\pi}{2}ar^2\,;\end{aligned}$$

thus the volume of the solid torus is $4\pi\cdot\dfrac{\pi}{2}ar^2 = 2\pi^2ar^2$ which agrees with what Example **??** shows.

Example 7.3. A solid D is formed by rotating the region bounded by the graph of $y = 1-\dfrac{x^2}{16}$ and $y=0$ about the x-axis. Then the volume of D computed by the disk method is given by

$$\pi\int_{-4}^{4}\Big(1-\frac{x^2}{16}\Big)^2\,dx = \pi\Big[x-\frac{x^3}{24}+\frac{x^5}{5\cdot 256}\Big]\Big|_{x=-4}^{x=4} = \frac{64\pi}{15}$$

while the volume of D computed by the shell method is given by

$$\begin{aligned}2\pi\int_0^1 y\big[\sqrt{16(1-y)}-(-\sqrt{16(1-y)})\big]\,dy &= 16\pi\int_0^1 y\sqrt{1-y}\,dy = 16\pi\int_1^0(1-u)u^{\frac{1}{2}}(-\,du)\\ &= 16\pi\int_0^1\big(u^{\frac{1}{2}}-u^{\frac{3}{2}}\big)\,du = 16\pi\Big[\frac{2}{3}u^{\frac{3}{2}}-\frac{2}{5}u^{\frac{5}{2}}\Big]_{u=0}^{u=1} = 16\pi\big(\frac{2}{3}-\frac{2}{5}\big) = \frac{64\pi}{15}\,.\end{aligned}$$

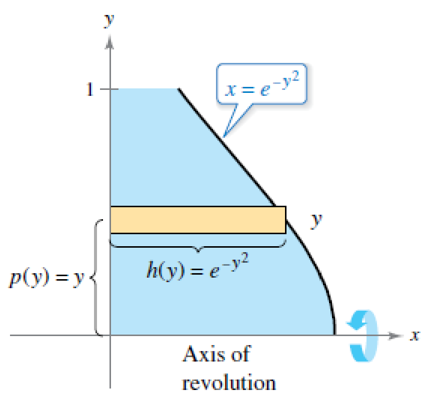
Now consider the volume of a solid D formed by revolving a region R about the line $y = L$, where

$$\mathrm{R} = \big\{(x,y)\,\big|\, y\in[c,d]\,, g(y)\leqslant x\leqslant f(y)\big\}$$

for some $c \geqslant L$ and continuous functions $f, g : [c, d] \to \mathbb{R}$ with $\min\limits_{y \in [c,d]} f(y) \geqslant \max\limits_{y \in [c,d]} g(y)$. Similar to the argument above, the volume of D is given by

$$2\pi \int_c^d (y - L)\big[f(y) - g(y)\big]\, dy\,.$$

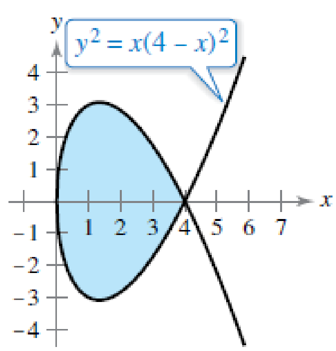
Example 7.4. Find the volume of the solid formed by revolving the region R about the x-axis, where R is the region bounded by the graph of $x = e^{-y^2}$, $y = 0$, $y = 1$ and the y-axis.



Using the shell method, the volume of this solid is given by

$$2\pi \int_0^1 (y - 0)e^{-y^2}\, dy = 2\pi\Big(-\frac{e^{-y^2}}{2}\Big)\Big|_{y=0}^{y=1} = \pi(1 - e^{-1})\,.$$

Example 7.5. Let R be the region bounded by the graph of $y^2 = x(4 - x)^2$.



Find the volume of the solid D formed by revolving R about

(a) the x-axis, (b) the y-axis, and (c) the line $x = 4$.

(a) Using the disk method, the volume of D is given by

$$\pi\int_0^4 x(4-x)^2\,dx = \pi\int_0^4 \big(x^3-8x^2+16x\big)\,dx = \pi\Big(\frac{1}{4}x^4-\frac{8}{3}x^3+8x^2\Big)\Big|_{x=0}^{x=4} = \frac{64\pi}{3}\,.$$

It will not be easy to compute the volume of D using the shell method since it requires solving for x (in terms of y) from $y^2 = x(4-x)^2$.

(b) Using the shell method, the volume of D is given by

$$\begin{aligned} 2\pi\int_0^4 x\big[\sqrt{x(4-x)^2}-(-\sqrt{x(4-x)^2})\big]\,dx &= 4\pi\int_0^4 x(4-x)x^{\frac{1}{2}}\,dx \\ = 4\pi\int_0^4 \big(4x^{\frac{3}{2}}-x^{\frac{5}{2}}\big)\,dx &= 4\pi\Big(\frac{8}{5}x^{\frac{5}{2}}-\frac{2}{7}x^{\frac{7}{2}}\Big)\Big|_{x=0}^{x=4} = \frac{2048\pi}{35}\,. \end{aligned}$$

(c) Using the shell method, the volume of D is given by

$$\begin{aligned} 2\pi\int_0^4 (4-x)\big[\sqrt{x(4-x)^2}-(-\sqrt{x(4-x)^2})\big]\,dx &= 4\pi\int_0^4 (x-4)^2x^{\frac{1}{2}}\,dx \\ = 4\pi\int_0^4 \big(x^{\frac{5}{2}}-8x^{\frac{3}{2}}+16x^{\frac{1}{2}}\big)\,dx &= 4\pi\Big(\frac{2}{7}x^{\frac{7}{2}}-\frac{16}{5}x^{\frac{5}{2}}+\frac{32}{3}x^{\frac{3}{2}}\Big)\Big|_{x=0}^{x=4} = \frac{8192\pi}{105}\,. \end{aligned}$$