Extra Exercise Problem Sets 6

Dec. 27 2018

Problem 1. In this exercise problem you are asked to show Stirling's formula

$$\lim_{n\to\infty} \frac{n!}{n^{n+0.5}e^{-n}} = \sqrt{2\pi}$$

through the following steps.

1. Show that the function $y = (1+x)^{\frac{1}{x}+0.5}$ is increasing on $\big(0, \frac{1}{2}\big)$.

   Hint: Show that the function $y = \big(\frac{1}{x} + \frac{1}{2}\big)\ln(1+x)$ is increasing on $\big(0, \frac{1}{2}\big)$ $\Big($you might want to show first that $\ln(1+x) \leqslant x - \frac{x^2}{2} + \frac{x^3}{3}$ for all $x \in (0,1)\Big)$ to conclude the monotonicity.

2. Let $s_n = \frac{n!}{n^{n+0.5}e^{-n}}$. Show that $s_n \geqslant s_{n+1} \geqslant 0$ for all $n \geqslant 2$. Therefore, the completeness of the real number implies that $\lim\limits_{n\to\infty} s_n = s$ exists (You do not have to show the red part. Just take it for granted if it is shown that $s_n \geqslant s_{n+1} \geqslant 0$ then the limit of s_n exists).

3. Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$. Show that $\lim\limits_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.

   Hint: Show that $\frac{I_{2n+2}}{I_{2n}} \leqslant \frac{I_{2n+1}}{I_{2n}} \leqslant 1$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \frac{I_{2n+2}}{I_{2n}} = 1$. You will need Wallis's formula.

4. Conclude that $\lim\limits_{n\to\infty} s_n = \sqrt{2\pi}$.