Extra Exercise Problem Sets 3

Oct. 19. 2018

Problem 1. Complete the following.

1. Show the Cauchy mean value theorem: Let $f, g : [a, b] \to \mathbb{R}$ be continuous and f, g are differentiable on (a, b). Show that if $g(a) \neq g(b)$ and $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in (a, b)$ such that
$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(c)}{g'(c)}\,.$$

2. Let $f, g : (a, b) \to \mathbb{R}$ be differentiable, $c \in (a, b)$, and $f(c) = g(c) = 0$. Show that if the limit $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists and
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}\,.$$
This is called L'Hôpital's rule.

3. Use the Cauchy mean value theorem to show that if $f : [a, b] \to \mathbb{R}$ is continuous and f is twice differentiable on (a, b) (that is, $f''(x)$ exists for all $x \in (a, b)$), then there exists $c \in (a, b)$ such that
$$f(b) = f(a) + f'(a)(b - a) + \frac{f''(c)}{2}(b - a)^2\,.$$