Calculus MA1001-A Midterm 2 Sample

National Central University, Dec. 17, 2018

Problem 1. 定理、定義敘述題。

Problem 2. Find $\dfrac{d}{dx}\displaystyle\int_{\ln x}^{\arctan x} 3^{-u^2}\,du$ for $x>0$. (Fundamental Theorem of Calculus)

Problem 3. Find $\displaystyle\lim_{x\to 0}\frac{1}{x}\exp\left(-\frac{1}{x^2}\right)$. (L'Hôspital's rule)

Problem 4. Find the following indefinite integrals: (integration by substitution)

1. $\displaystyle\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx.$ 2. $\displaystyle\int \frac{\sqrt{x}}{4+x^3}\,dx.$

Problem 5. Complete the following.

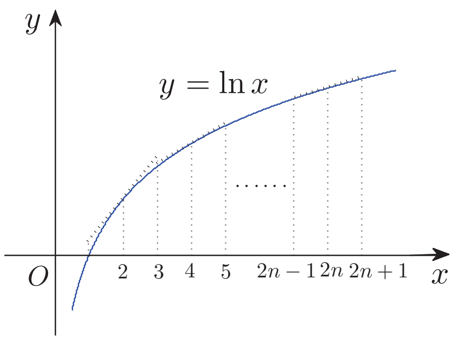
1. Let $0<a<c<b$. Prove that $\dfrac{x-c}{c}+\ln c \geqslant \ln x$ for all $a\leqslant x\leqslant b$. (the second derivative test)

   Hint: Define $f(x)=\ln x-\dfrac{x-c}{c}-\ln c$. Show that if $a<d<b$, $f(d)$ cannot be the maximum of f on $[a,b]$; thus the maximum of f on $[a,b]$ must be attained at the end-point a or b. Use this fact to conclude the desired inequality.

2. Interpret the inequality above geometrically (by drawing the figures of functions $y=\ln x$ and $y=\dfrac{x-c}{c}+\ln c$). (tangent lines of the graph of functions)

3. Show that $\left(n+\frac{1}{2}\right)\ln\left(n+\frac{1}{2}\right)+\frac{1}{2}\ln 2-n\leqslant \ln(n!)$ for all positive integers n. (the logarithmic properties of ln and the comparison of integrals)

   Hint: Using the conclusion in 1 and 2 to draw to following figure



   Evaluate the total area of these trapezoids and make use of the identity $\displaystyle\int_1^n \ln x\,dx = n\ln n - n + 1$.

Problem 6. Compute $\displaystyle\int \arcsin x\,dx$ by complete the following.

1. Use the substitution $x=\sin u$ to conclude that it suffices to compute the indefinite integral $\displaystyle\int x\cos x\,dx$. (integration by substitution)

2. Show that for $0 < x < \pi$,

$$\sum_{i=1}^{n} i\cos(ix) = \frac{-\cos\frac{x}{2}}{4\sin^2\frac{x}{2}}\Big[\cos\frac{x}{2} - \cos\big((n+\frac{1}{2})x\big)\Big] + \frac{1}{2\sin\frac{x}{2}}\Big[-\frac{1}{2}\sin\frac{x}{2} + (n+\frac{1}{2})\sin\big((n+\frac{1}{2})x\big)\Big]. \qquad (\star)$$

Hint: We have (more or less) shown in class that

$$\sum_{i=1}^{n} \sin(ix) = \frac{1}{2\sin\frac{x}{2}}\Big[\cos\frac{x}{2} - \cos\big((n+\frac{1}{2})x\big)\Big] \qquad \text{if} \quad \sin\frac{x}{2} \neq 0\,.$$

Differentiate the identity above to conclude $(\star)$. (differentiation)

3. Show that $\int_0^a x\cos x\,dx = a\sin a + \cos a - 1$ for all real number a by computing the limit of the Riemann sum of $y = x\cos x$ given by uniform partitions and the right end-point rule. (Riemann sum approximation of integrals)

4. Compute $\int \arcsin x\,dx$. (integration by substitution)